1: PHYS254: Electromagnetism 2006/07

THE UNIVERSITY
of LIVERPOOL

R-D Herzberg

1.1: Textbooks

- D.J Griffiths **Introduction to Electrodynamics** 3^{rd} ed. Prentice Hall
- W.J. Duffin **Electricity and Magnetism** 4^{th} ed. McGraw-Hill
- L.S. Grant and W.R. Phillips **Electromagnetism** 2^{nd} ed. Wiley

Supplements

- Schaum's Outlines: Vector Analysis *Murray R Spiegel*
- Schaum's Outlines: Electromagnetics 2^{nd} ed. *Joseph A Edminster*

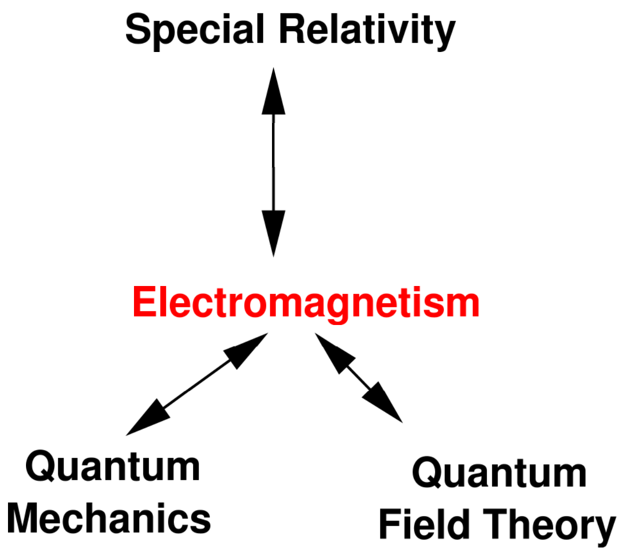
And many others.

1.2: Course Organisation

- 15 Credits
- 4 Tutorials
- 1 Class Test (Week 9)
- Marks: 80% for 3 hour exam
  20% for 1 Class test

Recommended private study time: **114 hours**
That's 9.5 hours per week or 2 hours per day Mo-Fr!

1.3: Introduction

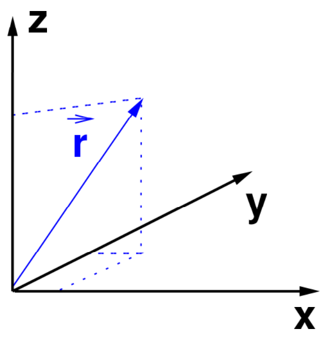


Electromagnetism is at the heart of modern physics. It has grown historically from the first ancient sailors using a floating piece of wood with a magnetic piece of ore on it via Faraday, Coulomb, Ampere and Ohm to James Clark Maxwell who unified *all* electric and magnetic effects into a system of 4 equations. We will follow his route to these equations.

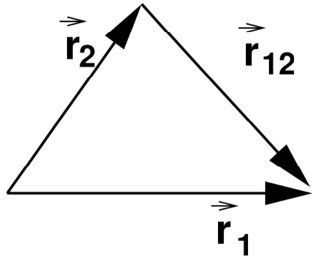
Electromagnetism is also tightly interconnected with Relativity, Quantum Mechanics, and modern field theory.

1.4: Reminder: Vectors

EM is formulated in vector form using differental equations. Thus we need to recapitulate vector calculus. None of this is new!



In 3-dimensional euclidian space we use a righthanded coordinate system x, y, z:
Each point in space is represented by a set of coordinates $\vec{r} = (x, y, z)$.



We also use vectors to describe distances, e.g from $\vec{r}_2$ to $\vec{r}_1$:
$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2 = (x_1, y_1, z_1) - (x_2, y_2, z_2) = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$

Each vector has a magnitude and a direction:

$$\vec{r} = |\vec{r}| \cdot \underline{\hat{\mathsf{a}}}_r$$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2} \qquad \text{Magnitude}$$

$\underline{\hat{\mathsf{a}}}_r$ Unit vector in the direction of $\vec{r}$

$$\underline{\hat{\mathsf{a}}}_r = \frac{\vec{r}}{|\vec{r}|} \qquad |\underline{\hat{\mathsf{a}}}_r| = 1$$

Useful unit vectors point along the coordinate axes:
$\underline{\hat{\mathsf{a}}}_x = (1, 0, 0)$ $\underline{\hat{\mathsf{a}}}_y = (0, 1, 0)$ $\underline{\hat{\mathsf{a}}}_z = (0, 0, 1)$

Thus we can write $\vec{r} = (x, y, z) = x \cdot \underline{\hat{\mathsf{a}}}_x + y \cdot \underline{\hat{\mathsf{a}}}_y + z \cdot \underline{\hat{\mathsf{a}}}_z$

There are many different notations in the literature, e.g. $\underline{\hat{\mathsf{a}}}_r \equiv \underline{\hat{\mathsf{r}}}$

Schaum uses $\underline{\hat{\mathsf{a}}}_x \equiv \underline{\hat{\mathsf{x}}} = \mathbf{i}$ $\underline{\hat{\mathsf{a}}}_y \equiv \underline{\hat{\mathsf{y}}} = \mathbf{j}$ $\underline{\hat{\mathsf{a}}}_z \equiv \underline{\hat{\mathsf{z}}} = \mathbf{k}$

These are too easily confused with current, imaginary unit and current density.

1.5: Example

The force of gravity a mass m_1 at $\vec{r}_1$ exerts on another mass m_2 at $\vec{r}_2$:

$$\vec{F}_{12} = G\frac{m_1 m_2}{r_{12}^2}\underline{\hat{\mathrm{r}}}_{12}$$

Alternatively:

$$\vec{F}_{12} = G\frac{m_1 m_2}{r_{12}^3}\vec{r}_{12}$$

You can use dimensional analysis if in doubt:
$[F]$ =N; $[m]$ =kg; $[r]$=m; $[G]$ =N m^2/kg^2

thus in the top example the units only match if $\underline{\hat{\mathrm{r}}}_{12}$ is a unit vector.

1.6: Vector Multiplication

Given two vectors $\vec{r}_1 = (x_1, y_1, z_1)$ and $\vec{r}_2 = (x_2, y_2, z_2)$ we can form two types of products:

Scalar product:

$$\vec{r}_1 \cdot \vec{r}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 = |\vec{r}_1||\vec{r}_2| \cdot \cos\vartheta$$

Vector Product:

$$\vec{r}_1 \times \vec{r}_2 = (x_1, y_1, z_1) \times (x_2, y_2, z_2) = (y_1 z_2 - y_2 z_1, z_1 x_2 - z_2 x_1, x_1 y_2 - x_2 y_1) =$$

$$(y_1 z_2 - y_2 z_1)\underline{\hat{a}}_x + (z_1 x_2 - z_2 x_1)\underline{\hat{a}}_y + (x_1 y_2 - x_2 y_1)\underline{\hat{a}}_z$$

The magnitude of the result is given by $|\vec{r}_1 \times \vec{r}_2| = |\vec{r}_1||\vec{r}_2| \cdot \sin\vartheta$ and its direction is perpendicular to both $\vec{r}_1$ and $\vec{r}_2$, such that $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_1 \times \vec{r}_2$ form a right-handed system.

2: Electrostatics

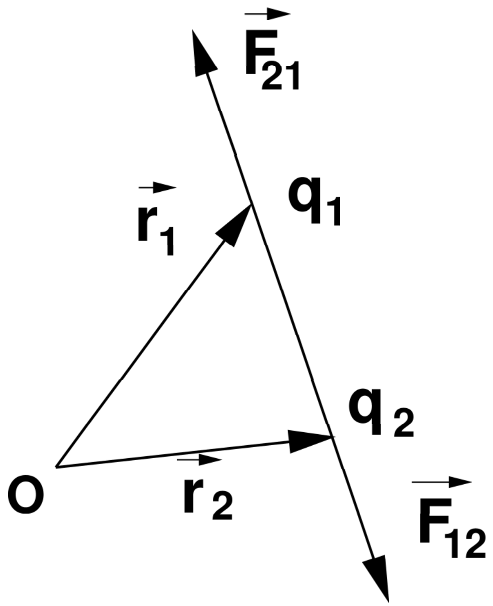
Revision: Coulomb's law

We start with a curious property of matter: charge. It comes in two flavors, positive and negative, and we know of their presence by detecting the force between them.

Charges are quantised in units of the elementary charge $e = 1.602 \times 10^{-19}$ C.

A lot of the course will deal with point charges denoted by q or Q. A point charge is assumed to be concentrated in a volume much smaller than the other dimensions of the problem, e.g. the distances between the point charges.

An electron is a point charge with a radius of less than 10^{-19} m!



We consider two charges q_1 and q_2 at points $\vec{r}_1$ and $\vec{r}_2$ in space. The force between them is experimentally determined to be proportional to the product of the two charges:

$$|\vec{F}_{12}| \sim q_1 \cdot q_2$$

It is also inversely proportional to the square of the distance between them:

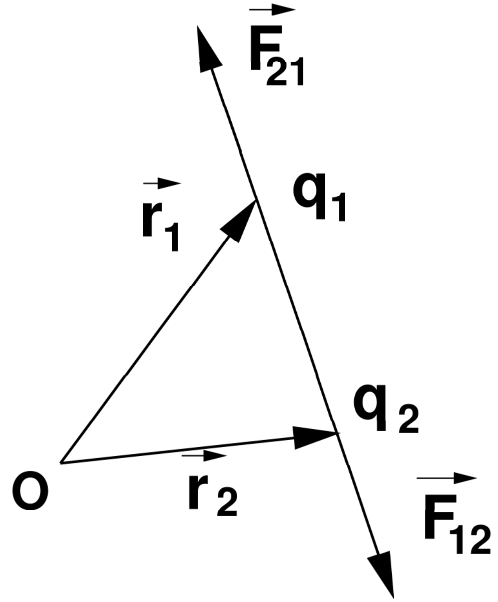
$$|\vec{F}_{12}| \sim \frac{1}{|\vec{r}_2 - \vec{r}_1|^2} = \frac{1}{|\vec{r}_{12}|^2}$$

This inverse square relationship has been verified to great precision in Cavendish's experiment. The force between charges q_1 and q_2 lies in the direction of the line through $\vec{r}_1$ and $\vec{r}_2$, i.e. parallel to $\vec{r}_2 - \vec{r}_1 = \vec{r}_{12}$:

$$\hat{\underline{\mathsf{F}}} = \pm \hat{\underline{\mathsf{r}}}_{12}$$

Thus we can write for the Force Charge 1 excerts on charge 2:

$$\vec{F}_{12} = K \cdot \frac{q_1 \cdot q_2}{|\vec{r}_2 - \vec{r}_1|^2} \hat{\underline{\mathsf{r}}}_{12} \qquad \text{or} \qquad \vec{F}_{12} = K \cdot \frac{q_1 \cdot q_2}{|\vec{r}_2 - \vec{r}_1|^3} (\vec{r}_2 - \vec{r}_1)$$



Clearly we have symmetry:

$$\vec{F}_{12} = -\vec{F}_{21}$$

The proportionality constant K is found through experiment. In SI units we have: $[q]$=C, $[\vec{F}]$=N, $[\vec{r}]$=m. Thus the dimension of K must be $[K]$=N m^2/C^2.

We find it convenient to write K in terms of the permittivity of free space ϵ_0

$$K = \frac{1}{4\pi\epsilon_0}$$

Today the Ampere is the SI base unit and ϵ_0 is defined to be exact in the SI system.

$$\epsilon_0 = 8.854 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\,m}^2} \quad \left[= \frac{\mathrm{C}^2\mathrm{s}^2}{\mathrm{kg\,m}^3} = \frac{\mathrm{F}}{\mathrm{m}} = \frac{\mathrm{A\,s}}{\mathrm{V\,m}} = \frac{\mathrm{C}}{\mathrm{V\,m}}\right]$$

2.1: Principle of Superposition

The Coulomb law obeys the principle of superposition: The force on a charge Q exerted by several other charges q_1 ... q_n is the vector sum of all forces exerted by each pair of charges Qq_i individually:

$$\begin{aligned} \vec{F}_Q &= \sum_{i=1}^{n} \vec{F}_{Qq_i} \\ &= \sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0} \frac{Qq_i}{|\vec{r}_Q - \vec{r}_{q_i}|^2} \hat{\underline{\mathrm{r}}}_{Qq_i} \\ &= \sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0} \frac{Qq_i}{|\vec{r}_Q - \vec{r}_{q_i}|^3} (\vec{r}_Q - \vec{r}_{q_i}) \end{aligned}$$

Sometimes we need to consider extended charge distributions, such as homogeneously charged plates. Then we define charge densities:

$$\rho(\vec{r}) = \quad \text{(Volume) charge density} \quad Q_{tot} = \int\limits_{Volume} \rho(\vec{r})\,\mathrm{d}\,V$$

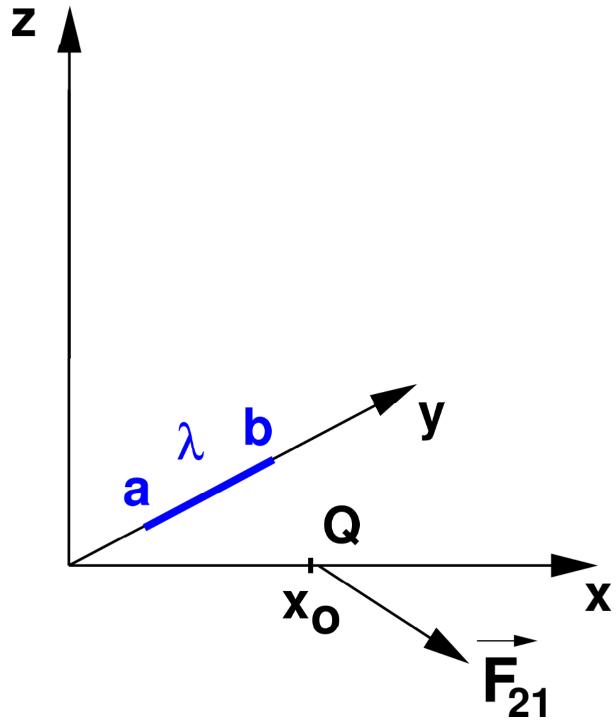
$$\sigma(\vec{r}) = \quad \text{Surface charge density} \quad Q_{tot} = \int\limits_{Area} \sigma(\vec{r})\,\mathrm{d}\,S$$

$$\lambda(\vec{r}) = \quad \text{Line charge density} \quad Q_{tot} = \int\limits_{Line} \lambda(\vec{r})\,\mathrm{d}\,l$$

The force on a charge Q exerted by a charge distribution $\rho\vec{r}$ in a volume V is given by integration:

$$\vec{F}_{Q\rho} = \int\limits_{V} \frac{Q}{4\pi\epsilon_0}\frac{\rho(\vec{r})}{|\vec{r}_Q - \vec{r}|^3}(\vec{r}_Q - \vec{r})\,\mathrm{d}\;V$$

2.2: Example



We will calculate the force on a charge Q exerted by a finite, straight, homogeneously charged line segment. The total charge q is distributed along the line. Thus the line charge density is $\lambda = q/(b-a)$.

$$\vec{F}_{Q\lambda} = \int_{Line} \frac{Q}{4\pi\epsilon_0} \frac{\lambda(\vec{r})}{|\vec{r}_Q - \vec{r}|^3} (\vec{r}_Q - \vec{r})\, \mathrm{d}\, l$$

The line runs from $\vec{r}_a = (0, a, 0)$ to $\vec{r}_b = (0, b, 0)$ along the y-axis. The charge Q is on the x-axis at $\vec{r}_Q = (x_0, 0, 0)$. This is the most general case. In cartesian coordinates the integral becomes:

$$\vec{F}_{Q\lambda} = \frac{Q\lambda}{4\pi\epsilon_0} \int_a^b \frac{(x_0, -y, 0)}{\left(\sqrt{x_0^2 + y^2}\right)^3} \mathrm{d}\, y$$

We can evaluate each component seperately: x first:

$$
\begin{aligned}
F_x &= \frac{Q\lambda}{4\pi\epsilon_0}\int_a^b \frac{x_0}{\left(\sqrt{x_0^2+y^2}\right)^3}\,\mathrm{d}\,y \\
&= \frac{Q\lambda}{4\pi\epsilon_0}\int_a^b \frac{x_0}{\left(x_0\sqrt{1+\left(\frac{y}{x_0}\right)^2}\right)^3}\,\mathrm{d}\,y \\
&= \frac{Q\lambda}{4\pi\epsilon_0}\int_\alpha^\beta \frac{x_0^2}{\cos^2\phi\left(x_0\sqrt{1+\tan^2\phi}\right)^3}\,\mathrm{d}\,\phi \\
&= \frac{Q\lambda}{4\pi\epsilon_0}\int_\alpha^\beta \frac{x_0^2\,\mathrm{d}\,\phi}{x_0^3\cos^2\phi\left(\sqrt{\frac{1}{\cos^2\phi}}\right)^3} \\
&= \frac{Q\lambda}{4\pi\epsilon_0}\frac{1}{x_0}\int_\alpha^\beta \cos\phi\,\mathrm{d}\,\phi \\
&= \frac{Q\lambda}{4\pi\epsilon_0}\left[\frac{\sin\phi}{x_0}\right]_\alpha^\beta
\end{aligned}
$$

substitute

$$
\begin{aligned}
\frac{y}{x_0} &= \tan\phi \\
\frac{\mathrm{d}\,y}{x_0} &= \frac{\mathrm{d}\,\phi}{\cos^2\phi} \\
\tan^2\phi+1 &= \frac{1}{\cos^2\phi}
\end{aligned}
$$

The limits are also straightforward:

$$
\begin{aligned}
\tan\alpha &= \frac{a}{x_0} \\
\sin\alpha &= \sqrt{1-\cos^2\alpha} = \sqrt{1-\frac{1}{1+\tan^2\alpha}} \\
&= \sqrt{\frac{\tan^2\alpha}{1+\tan^2\alpha}} \\
&= \frac{a}{\sqrt{x_0^2+a^2}} \\
\sin\beta &= \frac{b}{\sqrt{x_0^2+b^2}}
\end{aligned}
$$

Thus we get for the x-component of the integral:

$$
F_x = \frac{Q\lambda}{4\pi\epsilon_0}\int_a^b \frac{x_0}{\left(\sqrt{x_0^2+y^2}\right)^3}\mathrm{d}\,y = \frac{Q\lambda}{4\pi\epsilon_0 x_0}\left(\frac{b}{\sqrt{x_0^2+b^2}} - \frac{a}{\sqrt{x_0^2+a^2}}\right)
$$

The y-component is easier:

$$\begin{aligned}
F_y &= \frac{Q\lambda}{4\pi\epsilon_0}\int_a^b \frac{-y}{\left(\sqrt{x_0^2+y^2}\right)^3}\mathrm{d}\,y \\
&= \frac{Q\lambda}{4\pi\epsilon_0}\frac{(-1)}{2}\int_A^B u^{-3/2}\mathrm{d}\,u \\
&= \frac{Q\lambda}{4\pi\epsilon_0}\frac{(-1)}{2}\frac{(-2)}{1}\left[u^{-1/2}\right]_A^B \\
&= \frac{Q\lambda}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x_0^2+y^2}}\right]_a^b \\
&= \frac{Q\lambda}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x_0^2+b^2}} - \frac{1}{\sqrt{x_0^2+a^2}}\right]
\end{aligned}$$

substitute

$$u = x_0^2 + y^2$$

$$\mathrm{d}\,u = 2y\mathrm{d}\,y$$

The z-component is the easiest: $F_z = 0$. So the full result becomes:

$$\vec{F}_{Q\lambda} = \int\limits_{Line} \frac{Q}{4\pi\epsilon_0} \frac{\lambda(\vec{r})}{|\vec{r}_Q - \vec{r}|^3} (\vec{r}_Q - \vec{r}) \, \mathrm{d}\, l$$

$$\vec{F}_{Q\lambda} = \frac{Q\lambda}{4\pi\epsilon_0} \int_a^b \frac{(x_0, -y, 0)}{\left(\sqrt{x_0^2 + y^2}\right)^3} \mathrm{d}\, y$$

$$\vec{F}_{Q\lambda} = \frac{Q\lambda}{4\pi\epsilon_0} \left(\frac{1}{x_0} \left[\frac{b}{\sqrt{x_0^2 + b^2}} - \frac{a}{\sqrt{x_0^2 + a^2}} \right], \left[\frac{1}{\sqrt{x_0^2 + b^2}} - \frac{1}{\sqrt{x_0^2 + a^2}} \right], 0 \right)$$

As a special case we can look at the most symetrical case with $a = -b$: The result then becomes:

$$\vec{F}_{Q\lambda} = \frac{Q\lambda}{4\pi\epsilon_0} \left(\frac{1}{x_0} \left[\frac{b}{\sqrt{x_0^2 + b^2}} - \frac{-b}{\sqrt{x_0^2 + b^2}} \right], \left[\frac{1}{\sqrt{x_0^2 + b^2}} - \frac{1}{\sqrt{x_0^2 + b^2}} \right], 0 \right)$$

$$\vec{F}_{Q\lambda} = \frac{Q\lambda}{4\pi\epsilon_0} \left(\frac{1}{x_0} \left[\frac{2b}{\sqrt{x_0^2 + b^2}} \right], 0, 0 \right)$$

2.3: Summary

The Coulomb force on charge q_2 exerted by charge q_1:

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_2 \cdot q_1}{|\vec{r}_2 - \vec{r}_1|^2} \hat{\mathrm{r}}_{12} \qquad \text{or} \qquad \vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 \cdot q_2}{|\vec{r}_2 - \vec{r}_1|^3} (\vec{r}_2 - \vec{r}_1)$$

It obeys the principle of superposition: The force on a charge Q exerted by several other charges q_1 ... q_n is the vector sum of all forces exerted by each pair of charges Qq_i individually:

$$\vec{F}_Q = \sum_{i=1}^{n} \vec{F}_{q_i Q}$$

2.4: Mutual potential energy of point charges

Assume an empty universe except a single point charge q_1 at the origin of a freely defined coordinate system (and you, of course).

We bring another point charge q_2 from infinitely far away to its final position $\vec{r}$. At all times does charge q_2 feel a force due to q_1 and we have to use energy to move q_2 against this force.

Assume q_2 to move along the positive x-axis from infinity to x_0 with q_1 fixed at (0,0,0).

The Force is $$\vec{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0 x^2}\underline{\hat{\mathsf{a}}}_x$$

The energy needed to bring q_2 closer by dx is

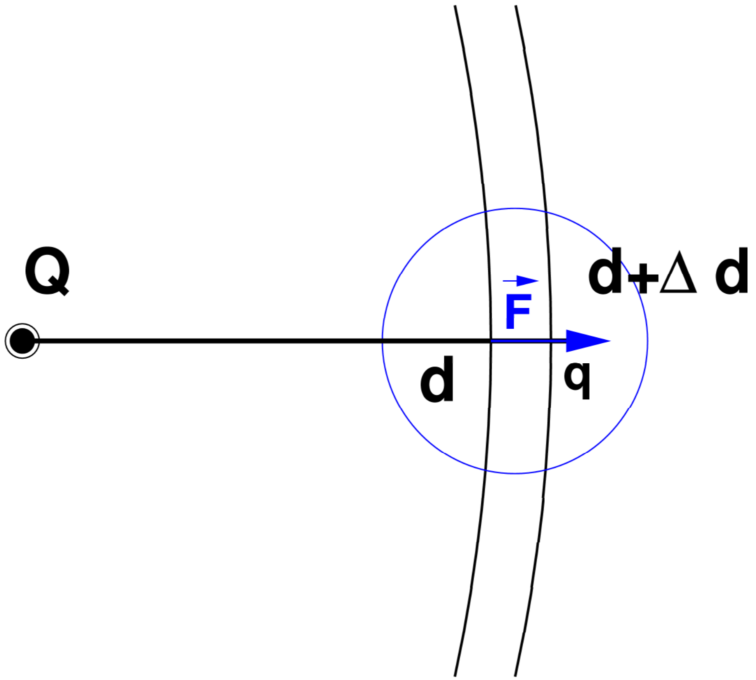
$$\mathrm{d}\,W = \vec{F}(x)\cdot\underline{\mathrm{d}\,x} = \vec{F}(x)(-\mathrm{d}\,x\,\underline{\hat{\mathsf{a}}}_x) = -\frac{q_1q_2}{4\pi\epsilon_0x^2}\mathrm{d}\,x\,\underline{\hat{\mathsf{a}}}_x\,\underline{\hat{\mathsf{a}}}_x$$

Total work

$$W = \int \mathrm{d}\,W = \int_\infty^{x_0} -\frac{q_1q_2}{4\pi\epsilon_0x^2}\mathrm{d}\,x$$

$$W = \left[\frac{+q_1q_2}{4\pi\epsilon_0x}\right]_\infty^{x_0} = \frac{q_1q_2}{4\pi\epsilon_0x_0}$$

All work has been converted to potential energy, thus we gained $\triangle U = W = \frac{q_1q_2}{4\pi\epsilon_0x_0}$ in the process.



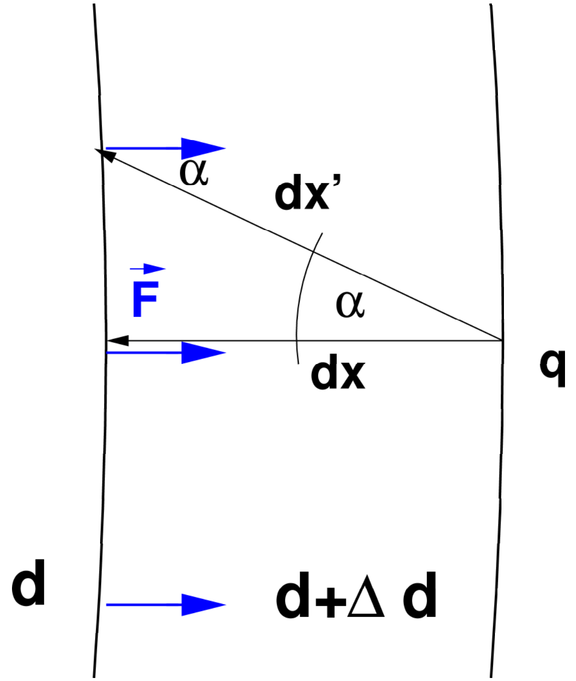
One last thing we have to show is that ΔU is independent of the path chosen.
Because of the principle of superposition it is enough to show the path is independent for 2 point charges

$$\vec{F} = \frac{Qq}{4\pi\epsilon_0 |r_{Qq}|^3} \vec{r}_{Qq}$$

This is a central force.

All points at a distance d experience the same magnitude of force. Thus we need to show that it does not matter how we move from one spherical shell to the next one and we are done.

$$\Delta U = \vec{F} \cdot \underline{\mathrm{d}\, x} = |\vec{F}|\, \mathrm{d}\, x \, \cos 180^\circ$$
$$\Delta U' = \vec{F} \cdot \underline{\mathrm{d}\, x'} = |\vec{F}|\, \mathrm{d}\, x' \cos (180^\circ - \alpha)$$



$$\frac{|\mathrm{d}\vec{x}|}{|\mathrm{d}\vec{x}'|} = \cos\alpha \quad \text{and} \quad \cos(180 - \alpha) = -\cos\alpha$$

$$\begin{aligned} \Rightarrow \Delta U &= -|\vec{F}|\mathrm{d}\, x \\ \Delta U' &= |\vec{F}|\mathrm{d}\, x' \cos(180 - \alpha) \\ &= -|\vec{F}|\mathrm{d}\, x \frac{\cos\alpha}{\cos\alpha} \\ &= \Delta U \end{aligned}$$

We say that the force is conservative.

What we calculated was a *change* in potential energy. Is there an absolute potential?

No, we are free to have additive constants, only changes in potential energy are important.
Here we found it convenient to use $U(r) \to 0$ as $r \to \infty$.

If we want to know the potential energy in a system of point charges we build up the final system by bringing the charges in from infinity one by one. This is possible because of the principle of superposition for the forces.

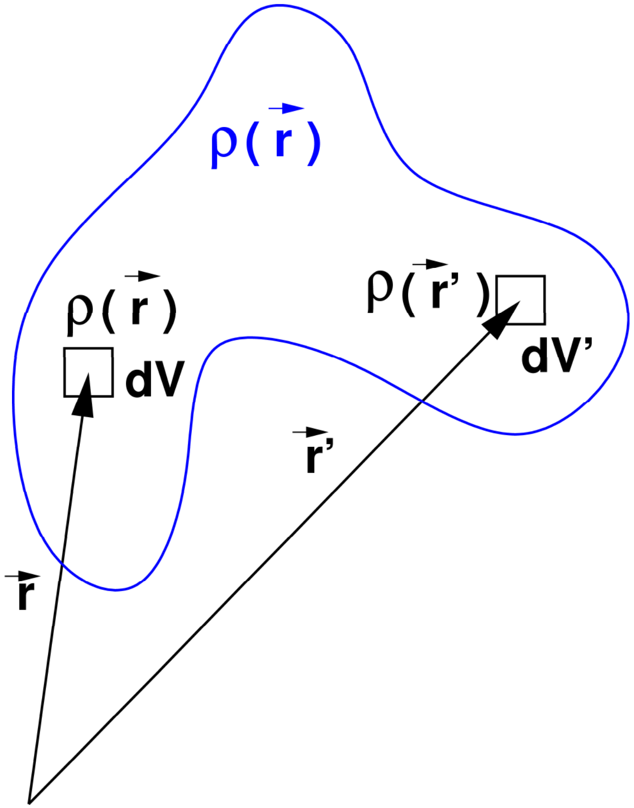
We get for 3 charges:

$$U = \frac{Q_1 Q_2}{4\pi\epsilon_0 |\vec{r}_{12}|} + \frac{Q_1 Q_3}{4\pi\epsilon_0 |\vec{r}_{13}|} + \frac{Q_2 Q_3}{4\pi\epsilon_0 |\vec{r}_{23}|}$$

or in general

$$U = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{Q_i Q_j}{4\pi\epsilon_0 |\vec{r}_{ij}|} = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{Q_i Q_j}{4\pi\epsilon_0 |\vec{r}_{ij}|}$$

($\frac{1}{2}$ to avoid double counting $Q_1 Q_2 = Q_2 Q_1$)



The latter form makes it easy to generalise to continuous charge distributions

The potential energy of the two infinitesimal charged volumes is

$$\mathrm{d}\,U = \frac{\rho(\vec{r})\mathrm{d}\,V \cdot \rho(\vec{r'})\mathrm{d}\,V'}{4\pi\epsilon_0|\vec{r'} - \vec{r}|}$$

And the total potential is given by

$$U = \frac{1}{2}\int_{V'}\int_{V}\frac{\rho(\vec{r})\rho(\vec{r'})}{4\pi\epsilon_0|\vec{r'} - \vec{r}|}\,\mathrm{d}\,V\mathrm{d}\,V'$$

The factor $\frac{1}{2}$ again avoids double counting.

2.5: Electric Field

The force between charges appears as a force at a distance with no medium carrying the force. We say that a charge Q fills space around it with an electric field, whose existence we measure by placing a positive test charge q at any point where we want to measure field. The field is defined as the force per unit charge.

$$\vec{E} = \frac{\vec{F}}{q} \qquad \text{parallel and proportional to } \vec{F}$$

Substituting Coulomb's law gives

$$\vec{E} = \frac{Q\, q\, \underline{\hat{\mathrm{r}}}_{Qq}}{4\pi\epsilon_0 |\vec{r}_{Qq}|^2} \cdot \frac{1}{q} = \frac{Q}{4\pi\epsilon_0 |\vec{r}_{Qq}|^2} \underline{\hat{\mathrm{r}}}_{Qq}$$

We can drop the indices to find the electric field due to a point charge Q at the origin of a coordinate system:

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 |\vec{r}|^2} \underline{\hat{\mathrm{r}}}$$

$\vec{E}$ is measured in $\frac{\mathrm{V}}{\mathrm{m}}$

2.6: Electric Potential

We were able to work out potential energy from forces. Remember from mechanics

$$\vec{F} = -\vec{\nabla} U \qquad \text{or} \qquad U = \int \vec{F} \vec{ds}$$

If the electric field is given via $\vec{E} = \vec{F}/q$ we can define an electric potential

$$V = \frac{U}{q}$$

for 2 point charges Q and q we had

$$U = \frac{Q_e q}{4\pi\epsilon_0 |\vec{r}_Q - \vec{r}_q|}$$

Now place Q at origin and find V

$$V(\vec{r}) = \frac{U(\vec{r})}{q} = \frac{Q}{4\pi\epsilon_0 |\vec{r}|}$$

From our definition we should now find

$$\vec{E} = -\vec{\nabla} V$$

This can easily be verified.

Remember $\vec{\nabla} \equiv grad \equiv (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

$\frac{Q}{4\pi\epsilon_0}$ is constant.

$$-\nabla V = \frac{-Q}{4\pi\epsilon_0}(\frac{\partial}{\partial x}\frac{1}{|\vec{r}|}, \frac{\partial}{\partial y}\frac{1}{|\vec{r}|}, \frac{\partial}{\partial z}\frac{1}{|\vec{r}|})$$

$$\frac{\partial}{\partial x}\frac{1}{|\vec{r}|} = \frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}} =$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot 2x = \frac{-x}{\sqrt{x^2+y^2+z^2}^3}$$

$$= \frac{-x}{|\vec{r}|^3}$$

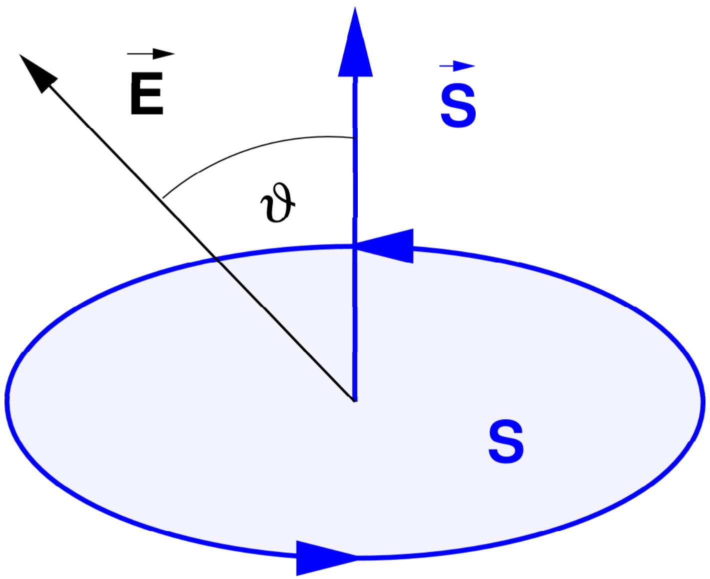
analog

$$\frac{\partial}{\partial y}\frac{1}{|\vec{r}|} = \frac{-y}{|\vec{r}|^3} \qquad \frac{\partial}{\partial z}\frac{1}{|\vec{r}|} = \frac{-z}{|\vec{r}|^3}$$

$$\rightarrow -\vec{\nabla}V = \frac{-Q}{4\pi\epsilon_0}\frac{(-x,-y,-z)}{|\vec{r}|^3} = \frac{Q}{4\pi\epsilon_0}\frac{\vec{r}}{|\vec{r}|^3} = \vec{E}$$

A word of caution: Although it is convenient to define $\vec{E}$ and V in terms of their mechanics counterparts, it turns out that $\vec{E}$ and V and have an independent life of their own. If $\vec{E}$ and V are given, then the forces and potential energy changes on real charges can be calculated from $\vec{F} = q\,\vec{E}$ and $\Delta U = q\Delta V$. Here $\vec{E}$ and V can be time dependent general fields *not* just produced by stationary point charges.

2.7: Revision: Gauss' Law / Electric Flux Φ



Assume a constant uniform electric field $\vec{E}$ passing through a surface S.

We can represent the surface as a vector of magnitude S and direction perpendicular to the surface $\rightarrow \vec{S}$.

The electric flux through S is

$$\Phi = |\vec{E}||\vec{S}| \cdot \cos\vartheta \qquad \text{or} \qquad \Phi = \vec{E}\,\vec{S}$$

If $\vec{E}$ varies or $\vec{S}$ is not plane we have to divide the surface into elements $\underline{\mathrm{d}\,S}$ sufficiently small so that over each surface element $\underline{\mathrm{d}\,S}$ $\vec{E}$ can be considered constant.

$$\text{Thus} \qquad \Phi = \int_S \vec{E}\underline{\mathrm{d}\,S}$$

We can choose the normal on either face of the surface, but if the surface is closed we always choose $\underline{\mathrm{d}\,S}$ to point out of the enclosed volume.

2.8: Gauss' Law for a point charge

Take a special case: A spherical surface of radius R around a point charge Q at its center.

The electric field is radial and has a constant magnitude on the sphere's surface. It is also perpendicular to the surface at all points. We can evaluate the flux

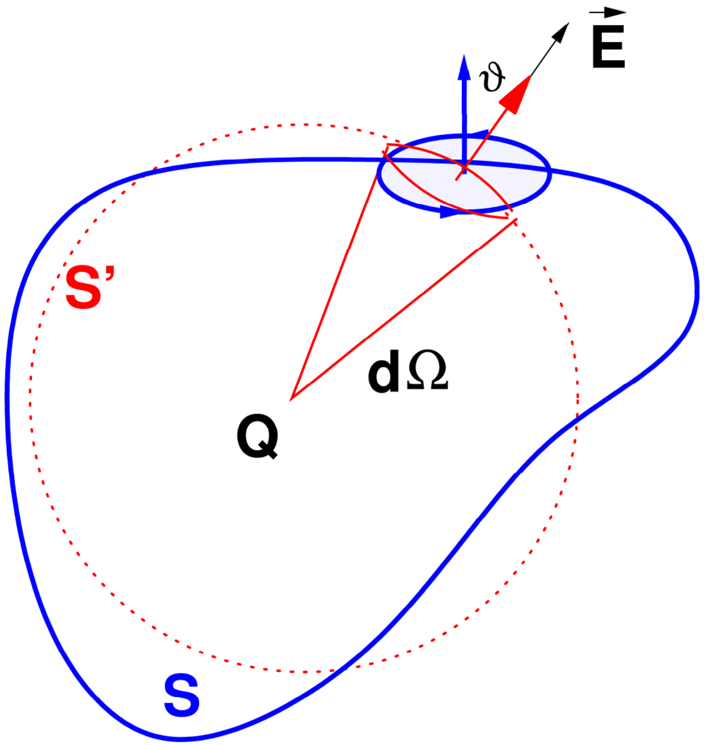
$$\Phi = |\vec{E}(R)| \cdot A = \frac{Q}{4\pi\epsilon_0 R^2} \cdot 4\pi R^2 = \frac{Q}{\epsilon_0}$$

We can get the same result without handwaving: Take a surface element $\underline{\mathrm{d}\,S}$ with an area dS and a direction radially outward: $\underline{\mathrm{d}\,S} = \mathrm{d}\,S\underline{\widehat{r}}$

$$\Phi = \int_{Sphere} \vec{E}\,\underline{\mathrm{d}\,S}$$

$$\vec{E} = \frac{Q\,\underline{\widehat{r}}}{4\pi\epsilon_0 R^2}$$

$$
\begin{aligned}
\rightarrow \Phi &= \int\limits_{Sphere} \frac{Q\,\underline{\widehat{r}}}{4\pi\epsilon_0 R^2} \cdot \underline{\widehat{r}}\,\mathrm{d}\,S \\
&= \int\limits_{Sphere} \frac{Q}{4\pi\epsilon_0 R^2}\,\mathrm{d}\,S \\
&= \frac{Q}{4\pi\epsilon_0 R^2} \int\limits_{Sphere} \mathrm{d}\,S \\
&= \frac{Q}{4\pi\epsilon_0 R^2}\,4\pi R^2 \\
&= \frac{Q}{\epsilon_0}
\end{aligned}
$$



We say "a charge Q gives rise to a flux $\frac{Q}{\epsilon_0}$". Indeed this is independent of the shape of the closed surface.

Consider element $\underline{\mathrm{d}\,S}$ on the general surface S we compare with the surface element $\mathrm{d}\,S'$ on a spherical surface through this point which covers the same solid angle $\mathrm{d}\,\Omega$.

Then $$\mathrm{d}\,\Omega = \frac{\mathrm{d}\,S'}{r^2} = \frac{\mathrm{d}\,S}{r^2}\cos\vartheta$$

Thus $$\begin{aligned} \vec{E}\,\underline{\mathrm{d}\,S} &= E r^2 \mathrm{d}\,\Omega \\ \vec{E}\,\underline{\mathrm{d}\,S} &= E\,\underline{\mathrm{d}\,S}\cos\vartheta = E\,r^2\mathrm{d}\,\Omega = \vec{E}\,\underline{\mathrm{d}\,S'} \end{aligned}$$

Thus we can rewrite the total flux out of a closed surface as

$$\begin{aligned} \Phi &= \int \vec{E}\,\underline{\mathrm{d}\,S'} = \int \vec{E}\,r^2\mathrm{d}\,\Omega = \int \frac{Q}{4\pi\epsilon_0}\mathrm{d}\,\Omega \\ &= \frac{Q}{4\pi\epsilon_0}\int \mathrm{d}\,\Omega = \frac{Q}{\epsilon_0} \end{aligned}$$

The principle of superposition translates directly to the fluxes. Thus the total flux through a closed surface is given by the sum of the individual fluxes due to all point charges in the enclosed volume.

$$\Phi = \sum_i \Phi_i = \sum \frac{Q_i}{\epsilon_0}$$

or

$$\oint \vec{E}\, \underline{\mathrm{d}\, S} = \sum \frac{Q_i}{\epsilon_0}$$

$\oint$ denotes an integral over a closed surface

This is the integral form of Gauss' law. It follows from Coulomb's law and the principle of superposition.

It allows us to calculate $\vec{E}$ for special symmetric situations

It allows to deduce the net charge in a bounded region of space given either the flux through the surface or the electric field given everywhere on the surface.

2.9: Applications: Charge distribution on a conductor

Conductor: charges are free to travel. We put a net charge Q onto an irregular conduction solid.

First we will show that no E field exists *inside* such a conductor. Then we show that no net charges exist inside either.

As long as an electric field exists charges will move. Eventually they reach an equilibrium state where no charge moves anymore. The E field must be zero everywhere inside the conductor. Moreover: $\vec{E} = -\vec{\nabla}V = 0$ means $V = const$ everywhere in the conductor, including the surface.

Now we start placing gaussian surfaces inside the conductor that do not enclose the surface of the conductor. The electric field on each gaussian surface is zero, thus the flux through it is also zero. That in turn shows that the gaussian surface does not enclose any net charge. All charge must accumulate on the surface.

2.10: Summary

The potential energy of a configuration of charges is given by

$$U = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{Q_i Q_j}{4\pi\epsilon_0 |\vec{r}_{ij}|} = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{Q_i Q_j}{4\pi\epsilon_0 |\vec{r}_{ij}|}$$

Or for continuous distributions as

$$U = \frac{1}{2} \int_{V'} \int_{V} \frac{\rho(\vec{r})\rho(\vec{r'})}{4\pi\epsilon_0 |\vec{r'} - \vec{r}|} \, \mathrm{d}\, V \mathrm{d}\, V'$$

($\frac{1}{2}$ to avoid double counting $Q_1 Q_2 = Q_2 Q_1$)

The electric field due to a point charge Q at the origin of a coordinate system is:

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 \vec{r}^2}\underline{\hat{\mathrm{r}}}$$

And the electric potential:

$$V(\vec{r}) = \frac{U(\vec{r})}{q} = \frac{Q}{4\pi\epsilon_0|\vec{r}|}$$

Potential and field are related via:

$$\vec{E} = -\vec{\nabla}V$$

Remember $\vec{\nabla} \equiv grad \equiv (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

The total flux through a closed surface is given by the sum of the individual fluxes due to all point charges in the enclosed volume.

$$\Phi = \sum_i \Phi_i = \sum \frac{Q_i}{\epsilon_0}$$

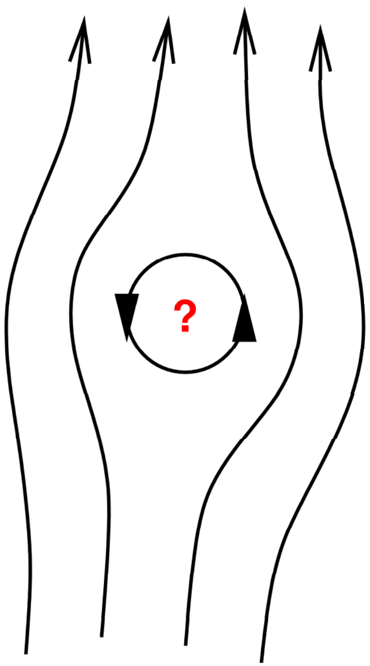
or

$$\oint \vec{E}\, \underline{\mathrm{d}\, S} = \sum \frac{Q_i}{\epsilon_0}$$

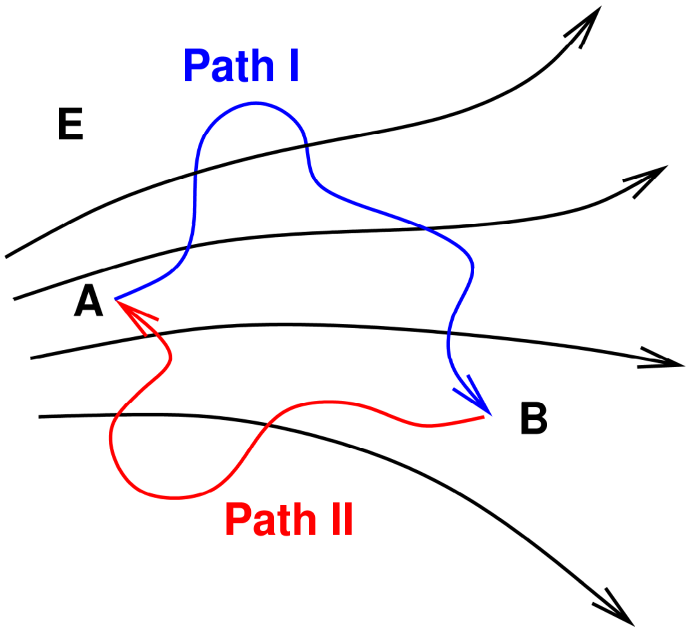
$\oint$ denotes an integral over a closed surface

No fields and charges exist inside a conductor

2.11: Revision: Circulation Law



Can an electrostatic field contain closed loops? Then we could evaluate the path integral $\oint \vec{E}\,\underline{\mathrm{d}\,l}$ around such a loop an get a numerical value different from zero, i.e. we could put a charge onto the loop and it would continually "ride round". This is in contradiction to the assumption of electrostatics. We shall prove this more formally:



Consider the closed loop from A along path I to B and back along path II. We have shown in our derivation of the electric potential that the path integral

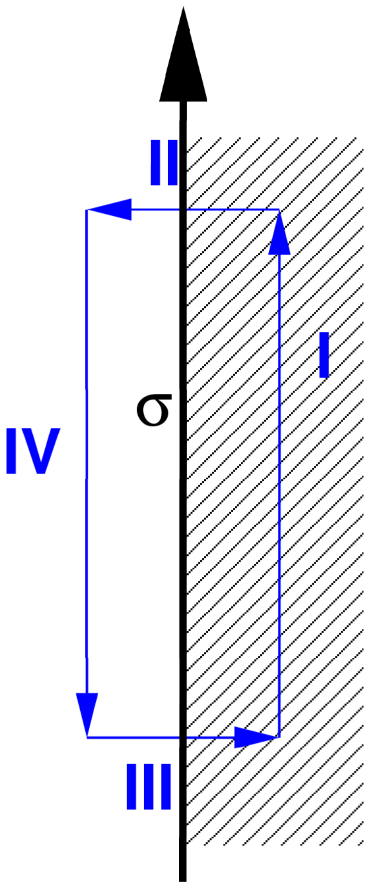
$$\int_{\vec{A}}^{\vec{B}} \vec{E}\, \underline{\mathrm{d}\, l} = V(\vec{B}) - V(\vec{A})$$

is independent of the path. More over

$$\int_{A}^{B} \vec{E}\, \underline{\mathrm{d}\, l} = -\int_{B}^{A} \vec{E}\, \underline{\mathrm{d}\, l}$$

is also independent of the path.
The integral around the entire loop is then

$$\begin{aligned} \oint \vec{E}\, \underline{\mathrm{d}\, l} &= \int_{A}^{B} \vec{E}\, \underline{\mathrm{d}\, l} + \int_{B}^{A} \vec{E}\, \underline{\mathrm{d}\, l} \\ &= V(\vec{B}) - V(\vec{A}) + V(\vec{A}) - V(\vec{B}) = 0 \end{aligned}$$

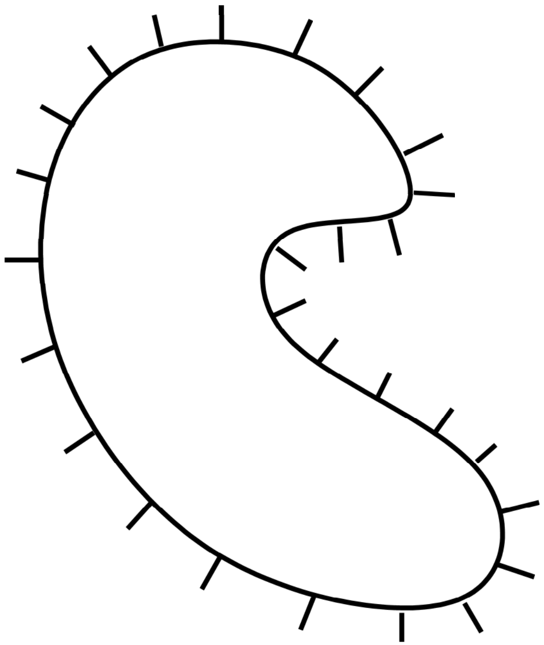


This argument only required an electrostatic situation, no assumptions were made about the shape of the path, it is therefore generally valid.
The circuital law for E in integral from is

$$\oint \vec{E}\,\underline{\mathrm{d}\,l} = 0$$

We can use this to learn something about the E field outside a charged conductor.
If we evaluate $\oint \vec{E}\,\underline{\mathrm{d}\,l}$ in sections we get no contribution from part I because $\vec{E}$ is zero inside a conductor.
We can make the contributions from part II and III as small as we want to by shortening the sections. That leaves part IV

$$\oint \vec{E}\,\underline{\mathrm{d}\,l} = \int_I \vec{E}\,\underline{\mathrm{d}\,l} + \int_{II} \vec{E}\,\underline{\mathrm{d}\,l} + \int_{III} \vec{E}\,\underline{\mathrm{d}\,l} + \int_{IV} \vec{E}\,\underline{\mathrm{d}\,l} = 0$$



$$\rightarrow \int_{IV} \vec{E}\, \underline{\mathrm{d}\, l} = 0$$

This means that $\vec{E}$ is perpendicular to $\underline{\mathrm{d}\, l}$ everywhere on the path IV. Since we chose the path to follow the surface we get a nice result:

The electric field outside a conducting solid enters (or leaves) the surface perpendicular to it !

Important: $\oint \vec{E}\, \underline{\mathrm{d}\, l} = 0$ only applies in electrostatics. Else we never have a working electric circuit.

2.12: Differential form of Gauss' Law

The $\vec{\nabla}$ operator is usually called 'del', in some texts you find it is called "nabla".

$$\vec{\nabla} \equiv \text{grad} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \qquad \text{in cartesian coordinates.}$$

We already know how to use it to form a gradient:

$$\begin{aligned} \vec{E} &= -\text{grad}V \\ \vec{E} &= -\vec{\nabla} V \qquad \text{e.g.} \qquad V = \frac{1}{|\vec{r}|} = \frac{1}{\sqrt{x^2+y^2+z^2}}. \end{aligned}$$

$$\begin{aligned} \vec{\nabla} V &= (\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}) = (\frac{-x}{\sqrt{x^2+y^2+z^2}^3}, \frac{-y}{\sqrt{x^2+y^2+z^2}^3}, \frac{-z}{\sqrt{x^2+y^2+z^2}^3}) \\ &= \frac{-r}{|\vec{r}|^3} = -\frac{1}{|\vec{r}|^2} \hat{\vec{r}} \end{aligned}$$

The gradient of a scalar field is a vector field. Its magnitude is the slope of the scalar field at any given point, and its direction points into the direction of steepest ascent.

We can also use $\vec{\nabla}$ to form two other differential operators on vector fields: divergence and curl.

Take a vector field $\vec{A}(\vec{r})$. You can form the divergence as

$$\begin{aligned} \text{div}\vec{A}(\vec{r}) &= \vec{\nabla} \cdot \vec{A}(\vec{r}) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (A_x(\vec{r}), A_y(\vec{r}), A_z(\vec{r})) \\ &= (\frac{\partial A_x(\vec{r})}{\partial x} + \frac{\partial A_y(\vec{r})}{\partial y} + \frac{\partial A_z(\vec{r})}{\partial z}) \end{aligned}$$

The divergence shows you the sources and drains of field lines at any point. E.g.:

$$\vec{A} = (x^2, yz, x)$$

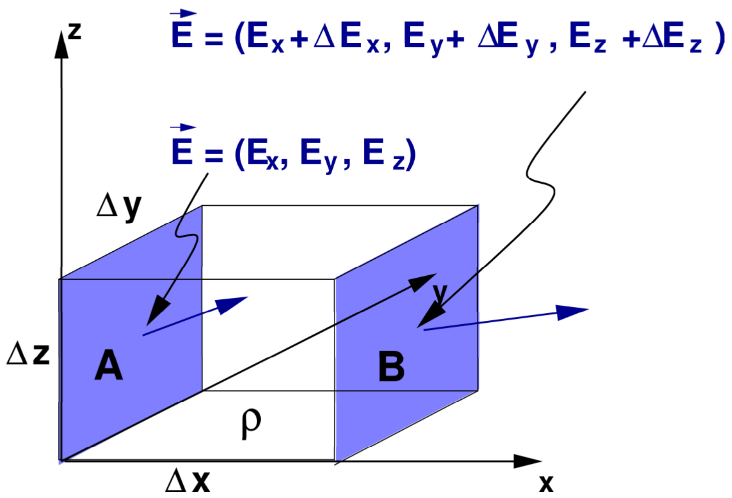
$$\text{div}\vec{A} = \nabla \cdot \vec{A} = \frac{\partial x^2}{\partial x} + \frac{\partial yz}{\partial y} + \frac{\partial x}{\partial z} = 2x + z$$

Second example:

$$\vec{A} = a\vec{r}$$

Calculate div $A = \vec{\nabla} \cdot \vec{A}$ yourself.

Now we rewrite Gauss' Law in differential form. Assume a small gaussian surface in a region of charge density $\rho(\vec{r})$. We make the surface a small box with sides parallel to a cartesian coordinate system of our chosing with sides $\Delta x, \Delta y, \Delta z$:



The flux through the shaded faces is $\vec{E}\mathrm{d}\vec{A}$. The normal vector on the face at A is

$$-\underline{\hat{a}}_x \Delta y \Delta z = (-\Delta y \Delta z, 0, 0)$$

and at B it is

$$\underline{\hat{a}}_x \Delta y \Delta z = (\Delta y \Delta z, 0, 0)$$

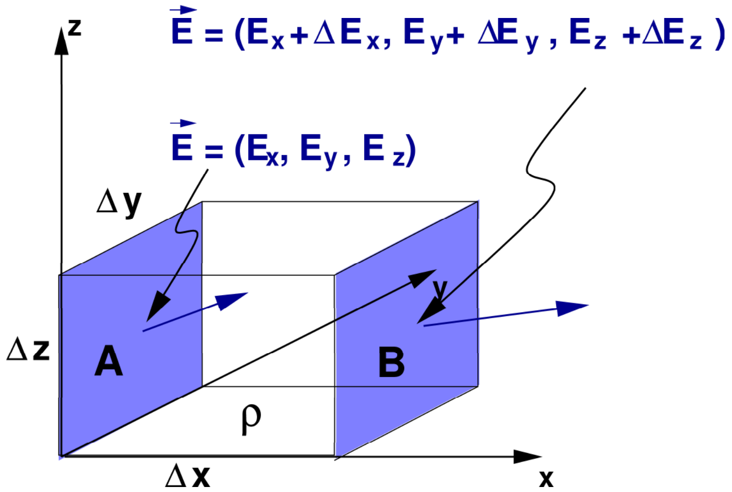
Thus the flux through these two faces is

$$\begin{aligned}
\Phi_{\text{Shad}} &= (E_x, E_y, E_z) \cdot (-\Delta y \Delta z, 0, 0) \\
&+ (E_x + \Delta E_x, E_y + \Delta E_y, E_z + \Delta E_z) \cdot (\Delta y \Delta z, 0, 0) \\
&= -E_x \Delta y \Delta z + E_x \Delta y \Delta z + \Delta E_x \Delta y \Delta z \\
&= \Delta E_x \Delta y \Delta z
\end{aligned}$$

ΔE_x can be written to first order as $\frac{\partial E_x}{\partial x}\Delta x$

$$\rightarrow \Phi_{Shad} = \frac{\partial E_x}{\partial x}\Delta x \Delta y \Delta z$$

Similar arguments go for the other two faces to give the total flux



$$\Phi = \left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right]\Delta x \Delta y \Delta z$$

The right hand side of Gauss' Law is $\quad Q/\epsilon_0$

$$\Phi = \frac{Q}{\epsilon_0} = \frac{\rho(\vec{r})\Delta x \Delta y \Delta z}{\epsilon_0}$$

As we now shrink $\Delta x, \Delta y, \Delta z$ to zero we are left with

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0}$$

or

$$\text{div}\vec{E} = \frac{\rho(\vec{r})}{\epsilon_0} \qquad \text{or} \qquad \vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}$$

This is a general result and one of the fundamental equations of nature.

$$\text{div}\vec{E} = \frac{\rho(\vec{r})}{\epsilon_0} \qquad \text{or} \qquad \vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}$$

We will later recognize this as one of Maxwell's equations.

It no longer depends on a choice of surface, but gives us the *local* behaviour of the electric field.

Gauss' law in differential form can be stated as

"The sources and drains of the electric field are the charges."

Another example: Consider the electric field of a point charge $\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{\vec{r}}{|\vec{r}|^3}$

$$\begin{aligned}
\text{div}\vec{E} &= \vec{\nabla}\cdot\vec{E} \\
&= \frac{Q}{4\pi\epsilon_0}\Big[\frac{\partial}{\partial x}\frac{x}{(\sqrt{x^2+y^2+z^2})^3} + \frac{\partial}{\partial y}\frac{y}{(\sqrt{x^2+y^2+z^2})^3} + \frac{\partial}{\partial z}\frac{z}{(\sqrt{x^2+y^2+z^2})^3}\Big]
\end{aligned}$$

This has a pole at $\vec{r} = 0$. So calculate $\vec{\nabla}\cdot\vec{E}$ for $(x, y, z) \neq (0, 0, 0)$:

First take the derivative with respect to x:

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{x}{(\sqrt{x^2+y^2+z^2})^3} &= \frac{\sqrt{x^2+y^2+z^2}^3 - x\frac{3}{2}\sqrt{x^2+y^2+z^2}\cdot 2\,x}{(\sqrt{x^2+y^2+z^2})^6} \\
&= \frac{x^2+y^2+z^2-3x^2}{(\sqrt{x^2+y^2+z^2})^5} \\
&= \frac{-2x^2+y^2+z^2}{(\sqrt{x^2+y^2+z^2})^5}
\end{aligned}$$

The derivatives w.r.t. y and z are completely analogous:

$$\frac{\partial}{\partial y}\frac{y}{(\sqrt{x^2+y^2+z^2})^3}=\frac{x^2-2y^2+z^2}{(\sqrt{x^2+y^2+z^2})^5}$$

$$\frac{\partial}{\partial z}\frac{z}{(\sqrt{x^2+y^2+z^2})^3}=\frac{x^2+y^2-2z^2}{(\sqrt{x^2+y^2+z^2})^5}$$

Bring them all together

$$\rightarrow \vec{\nabla}\cdot\vec{E}=\frac{Q}{4\pi\epsilon_0}\frac{(-2x^2+y^2+z^2+x^2-2y^2+z^2+x^2+y^2-2z^2)}{(\sqrt{x^2+y^2+z^2})^5}=0$$

At $\vec{r}=\vec{0}$ we have the point charge and the electric field has a pole. We can use Gauss' law to find $\vec{\nabla}\cdot\vec{E}$ at the origin.

The charge density is zero everywhere except at the origin, where it is concentrated in a point:

The charge density is

$$\rho(\vec{r}) = Q\delta(\vec{r})$$

and we can write

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}$$

Remember: the δ function is defined as

$$\delta(\vec{r}) = 0 \ \text{ for } \vec{r} \neq 0$$

$$\delta(\vec{r}) = \infty \ \text{ for } \vec{r} = 0$$

and

$$\int_V \delta(\vec{r})\, dV = 1$$

2.13: Summary

The circuital law in integral form is:

$$\oint \vec{E}\, \underline{\mathrm{d}\, l} = 0$$

The electric field inside any conductor is zero.

The electric field outside a conducting solid enters (or leaves) the surface perpendicular to it !

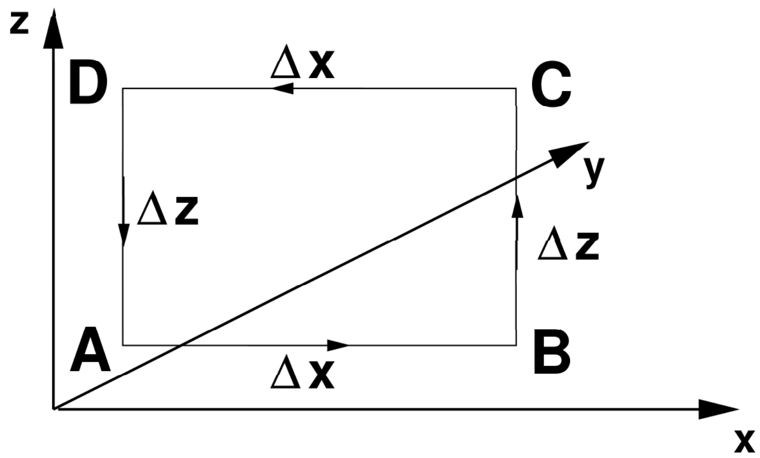
Gauss' law in differential form is:

$$\mathrm{div}\vec{E} = \frac{\rho(\vec{r})}{\epsilon_0} \qquad \text{or} \qquad \vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}$$

"The sources and drains of the electric field are the charges."

2.14: The circuital law in differential form

We now look at the circuital law



The electric field of the corners of this loop is:

$$
\begin{aligned}
\vec{E}_A &= (E_x, E_y, E_z) \\
\vec{E}_B &= (E_x + \frac{\partial E_x}{\partial x}\Delta x, E_y + \frac{\partial E_y}{\partial x}\Delta x, E_z + \frac{\partial E_z}{\partial x}\Delta x) \\
\vec{E}_C &= (E_x + \frac{\partial E_x}{\partial x}\Delta x + \frac{\partial E_x}{\partial z}\Delta z, E_y + \frac{\partial E_y}{\partial x}\Delta x + \frac{\partial E_y}{\partial z}\Delta z, E_z + \frac{\partial E_z}{\partial x}\Delta x + \frac{\partial E_z}{\partial z}\Delta z) \\
\vec{E}_D &= (E_x + \frac{\partial E_x}{\partial z}\Delta z, E_y + \frac{\partial E_y}{\partial z}\Delta z, E_z + \frac{\partial E_z}{\partial z}\Delta z)
\end{aligned}
$$

We take an average field at the centre of the side for our calculation.

$$
\begin{array}{llcll}
A \to B & \mathrm{d}\vec{l} & = & \Delta x\,\underline{\hat{\mathsf{a}}}_x & \Rightarrow \quad \vec{E}\,\mathrm{d}\vec{l} = E_x\,\Delta x + \frac{1}{2}\frac{\partial E_x}{\partial x}\,(\Delta x)^2 \\
B \to C & \mathrm{d}\vec{l} & = & \Delta z\,\underline{\hat{\mathsf{a}}}_z & \Rightarrow \quad \vec{E}\,\mathrm{d}\vec{l} = E_z\,\Delta z + \frac{\partial E_z}{\partial x}\Delta x\Delta z + \frac{1}{2}\frac{\partial E_z}{\partial z}\,(\Delta z)^2 \\
C \to D & \mathrm{d}\vec{l} & = & -\Delta x\,\underline{\hat{\mathsf{a}}}_x & \Rightarrow \quad \vec{E}\,\mathrm{d}\vec{l} = -E_x\,\Delta x - \frac{\partial E_x}{\partial z}\Delta z\Delta x - \frac{1}{2}\frac{\partial E_x}{\partial x}\,(\Delta x)^2 \\
D \to A & \mathrm{d}\vec{l} & = & -\Delta z\,\underline{\hat{\mathsf{a}}}_z & \Rightarrow \quad \vec{E}\,\mathrm{d}\vec{l} = -E_z\,\Delta z - \frac{1}{2}\frac{\partial E_z}{\partial z}\,(\Delta z)^2
\end{array}
$$

$$
\begin{aligned}
\to \oint \vec{E}\mathrm{d}\vec{l} &= \frac{\partial E_z}{\partial x}\Delta x\Delta z - \frac{\partial E_x}{\partial z}\Delta z\Delta x \\
&= (\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z})\Delta z\Delta x = 0
\end{aligned}
$$

$$
\to \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = 0
$$

Analog for paths in the x-y plane

$$\rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$$

and the y-z plane

$$\rightarrow \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0$$

We can rewrite these three results with the curl operator

$$\text{curl}\, \vec{E} = \vec{0} \qquad \text{or} \qquad \vec{\nabla} \times \vec{E} = \vec{0}$$

$$(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (E_x, E_y, E_z) = (0,0,0)$$

$$= \begin{vmatrix} \underline{\hat{a}}_x & \underline{\hat{a}}_y & \underline{\hat{a}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = (\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) = (0,0,0)$$

An electric field created solely from static charges has no closed loops. Its curl is zero everywhere.

E.g.: $$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{\vec{r}}{|\vec{r}|^3}$$

$$\vec{\nabla} \times \vec{E} = \frac{Q}{4\pi\epsilon_0}\left(\frac{\partial}{\partial y}\frac{z}{(\sqrt{x^2+y^2+z^2})^3} - \frac{\partial}{\partial z}\frac{y}{(\sqrt{x^2+y^2+z^2})^3}, \frac{\partial}{\partial z}\frac{x}{(\sqrt{x^2+y^2+z^2})^3} - \frac{\partial}{\partial x}\frac{z}{(\sqrt{x^2+y^2+z^2})^3}, \frac{\partial}{\partial x}\frac{y}{(\sqrt{x^2+y^2+z^2})^3} - \frac{\partial}{\partial y}\frac{x}{(\sqrt{x^2+y^2+z^2})^3}\right)$$

$$\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right) = \frac{-z \cdot \frac{3}{2}\sqrt{x^2+y^2+z^2} \cdot 2y}{(\sqrt{x^2+y^2+z^2})^6}$$

$$\frac{\partial}{\partial z}\left(\frac{y}{r^3}\right) = \frac{-y \cdot \frac{3}{2}\sqrt{x^2+y^2+z^2} \cdot 2z}{(\sqrt{x^2+y^2+z^2})^6} \quad \rightarrow \quad \vec{\nabla} \times \vec{E} = 0$$

2.15: Examples

Take a vector field $\vec{F} = F_0(-y, x, 1)$.

$$\text{curl}\,\vec{F} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times F_0(-y, x, 1)$$

$$= \begin{vmatrix} \underline{\hat{\mathsf{a}}}_x & \underline{\hat{\mathsf{a}}}_y & \underline{\hat{\mathsf{a}}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -F_0 y & F_0 x & F_0 \end{vmatrix} = (\frac{\partial F_0}{\partial y} - \frac{\partial F_0 x}{\partial z}, -\frac{\partial F_0 y}{\partial z} - \frac{\partial F_0}{\partial x}, \frac{\partial F_0 x}{\partial x} + \frac{\partial (F_0 y)}{\partial y}) = (0, 0, 2F_0)$$

Take any continuous differentiable scalar field $\Phi(\vec{r})$.

Calculate curl grad $\Phi(\vec{r})$

$$\text{curl grad}\Phi(\vec{r}) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z})$$

$$= \begin{vmatrix} \underline{\hat{\mathsf{a}}}_x & \underline{\hat{\mathsf{a}}}_y & \underline{\hat{\mathsf{a}}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} & \frac{\partial \Phi}{\partial z} \end{vmatrix} = (\frac{\partial^2 \Phi}{\partial y \partial z} - \frac{\partial^2 \Phi}{\partial z \partial y}, \frac{\partial^2 \Phi}{\partial z \partial x} - \frac{\partial^2 \Phi}{\partial x \partial z}, \frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\partial^2 \Phi}{\partial y \partial x}) = (0, 0, 0)$$

Thus: curl grad $\Phi = 0$.

Other identities:

Show that for any continuous differentiable vector field $\vec{F}(\vec{r})$ we have

$$\text{div curl}\vec{F} = 0$$

A general rule for vectors is the bac-cab rule:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Try this with $\vec{\nabla}$:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{F}(\vec{r})) = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{F}$$

2.16: Poisson and Laplace Equations

We had Gauss' Law in differential form

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

And we found $\vec{E}$ from a potential V via

$$\vec{E} = -\vec{\nabla} V$$

This gives

$$\vec{\nabla} \cdot (-\vec{\nabla} V) = \frac{\rho}{\epsilon_0} \qquad \text{or} \qquad \vec{\nabla} \cdot (\vec{\nabla} V) = -\frac{\rho}{\epsilon_0}$$

$$\text{div grad} V = -\frac{\rho}{\epsilon_0}$$

$$(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}) = -\frac{\rho}{\epsilon_0}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon_0}$$

We can define the Laplace Operator $\nabla^2 = (\vec{\nabla} \cdot \vec{\nabla})$ in cartesian coordinates:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Thus we arrive at Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

At points where no charges exist, this reduces to Laplace's equation

$$\nabla^2 V = 0$$

Let's see how we can work with them.

First find the charge distribution from a given potential:

Take $\quad V = V_0 \exp\left(-\left(\frac{r}{a}\right)^2\right)$

$$\nabla^2 V = V_0 \left[\frac{\partial^2}{\partial x^2} \exp\left(-\left(\frac{r}{a}\right)^2\right) + \frac{\partial^2}{\partial y^2} \exp\left(-\left(\frac{r}{a}\right)^2\right) + \frac{\partial^2}{\partial z^2} \exp\left(-\left(\frac{r}{a}\right)^2\right)\right]$$

$$\begin{aligned}
\frac{\partial}{\partial x} \exp\left(-\left(\frac{r}{a}\right)^2\right) &= -2\frac{r}{a^2} \exp\left(-\left(\frac{r}{a}\right)^2\right) \cdot \frac{x}{r} \\
\frac{\partial^2}{\partial x^2} \exp\left(-\left(\frac{r}{a}\right)^2\right) &= \frac{\partial}{\partial x} \frac{-2x}{a^2} \exp\left(-\left(\frac{r}{a}\right)^2\right) \\
&= \frac{-2x}{a^2} \cdot \exp\left(-\left(\frac{r}{a}\right)^2\right) \cdot \frac{-2x}{a^2} - \frac{2}{a^2} \exp\left(-\left(\frac{r}{a}\right)^2\right) \\
&= \left(\frac{4x^2}{a^4} - \frac{2}{a^2}\right) \exp\left(-\left(\frac{r}{a}\right)^2\right)
\end{aligned}$$

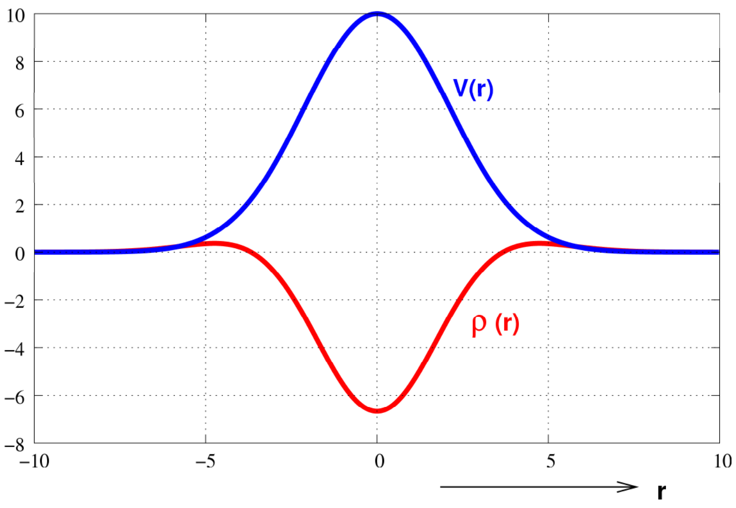
Similarly you get

$$\frac{\partial^2 V}{\partial y^2} = (\frac{4y^2}{a^4} - \frac{2}{a^2}) \exp\left(-\left(\frac{r}{a}\right)^2\right)$$

and

$$\frac{\partial^2 V}{\partial z^2} = (\frac{4z^2}{a^4} - \frac{2}{a^2}) \exp\left(-\left(\frac{r}{a}\right)^2\right)$$

Thus the result is



$$\begin{aligned} \nabla^2 V &= V_0(\frac{4}{a^4}(x^2 + y^2 + z^2) - \frac{3 \cdot 2}{a^2}) \exp\left(-\left(\frac{r}{a}\right)^2\right) \\ &= V_0(\frac{4r^2}{a^4} - \frac{6}{a^2}) \exp\left(-\left(\frac{r}{a}\right)^2\right) \\ &= -\frac{\rho(\vec{r})}{\epsilon_0} \end{aligned}$$

The Laplace operator can act on a scalar field Φ as well as a vector field $\vec{F}$:

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial\, x^2} + \frac{\partial^2 \Phi}{\partial\, y^2} + \frac{\partial^2 \Phi}{\partial\, z^2}$$

$$\begin{aligned} \nabla^2 \vec{F} &= \left(\nabla^2 F_x\,,\, \nabla^2 F_y\,,\, \nabla^2 F_z\right) \\ &= \left(\frac{\partial^2 F_x}{\partial\, x^2} + \frac{\partial^2 F_x}{\partial\, y^2} + \frac{\partial^2 F_x}{\partial\, z^2},\quad \frac{\partial^2 F_y}{\partial\, x^2} + \frac{\partial^2 F_y}{\partial\, y^2} + \frac{\partial^2 F_y}{\partial\, z^2},\quad \frac{\partial^2 F_z}{\partial\, x^2} + \frac{\partial^2 F_z}{\partial\, y^2} + \frac{\partial^2 F_z}{\partial\, z^2}\right) \end{aligned}$$

It is easy to find the charge distribution given the potential. It is more tricky to find the potential given the charge distribution.

Can we do it? Not yet. The problem is not yet completely defined. The potential can be found by integrating the charge density twice. But in each integration you get a degree of freedom that needs to be pinned down.

Imagine that you want to work out the electric potential in a room. Poisson's equation is valid everywhere in the room.

Now assume the room had a conducting wallpaper. That would be an equipotential surface and the electric field would have to be normal to the wallpaper. If the same wallpaper were an insulator, the field configuration could look very different.

We need to define the boundary conditions!

Once they are specified we will find that the Poisson and Laplace equations have only one unique solution:

Consider an enclosed region of space. Let the potential be fixed everywhere on the boundary. We allow charges inside the volume so Poisson's equation is valid.

We will now show that Poisson's equation has a unique solution by assuming it has two distinct solutions V_1 and V_2 and then showing that $V_1 = V_2$ everywhere.

Both V_1 and V_2 obey Poisson's equation

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon_0} \qquad \text{and} \qquad \nabla^2 V_2 = -\frac{\rho}{\epsilon_0}$$

On the boundary we have $V_1 = V_2$, to fulfill the boundary conditions. Now take a potential $V_3 = V_1 - V_2$

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{\rho}{\epsilon_0} - (-\frac{\rho}{\epsilon_0}) = 0$$

V_3 fulfills Laplace's equation inside the volume.

$$\nabla^2 V_3 = 0$$

V_3 thus acts in the enclosed volume *as if* no charges were present. Furthermore, on the boundary $V_3 = V_1 - V_2 = 0$. The volume is bounded by an equipotential surface .

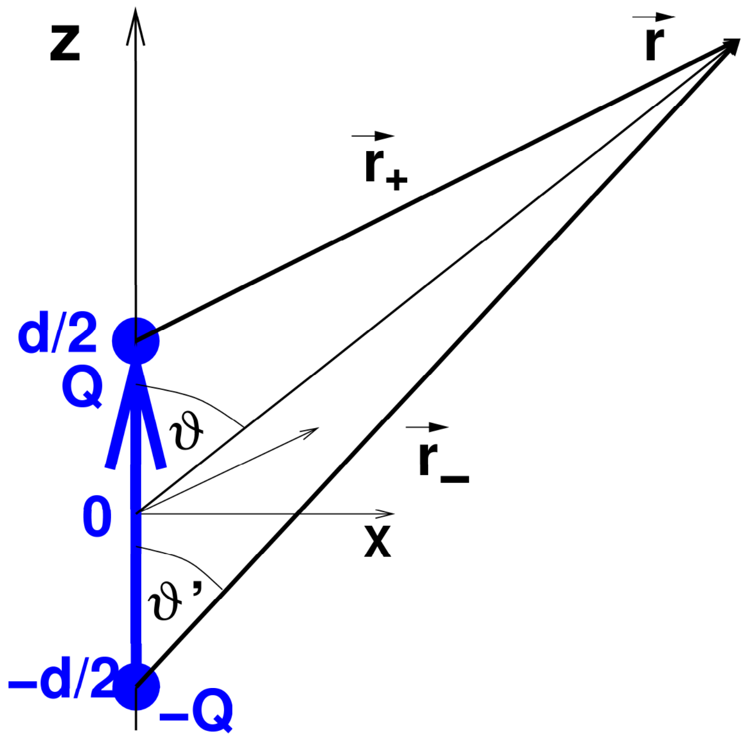
Earlier we showed that a region of space containing no charges bound by an equipotential surface contains no electric field, thus the electric potential was constant everywhere within that volume.

V_3 is a potential in such a region. It must be constant inside and we know it is zero at the boundary. Therefore it must be zero everywhere.

We get $V_1 - V_2 = V_3 = 0$ or $V_1 = V_2$ in contradiction to our assumption that two distinct potentials exist as solutions of Poisson's equation.

Possion's equation has a unique solution if the boundary conditions are sufficiently well specified.

2.17: Application: The electric Dipole



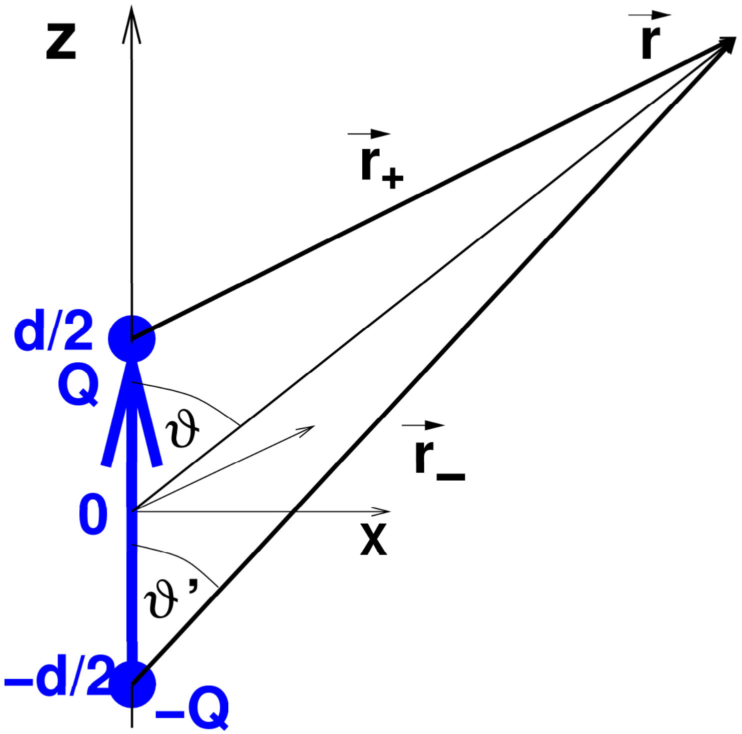
Consider two equal and opposite charges $q_1 = Q$, $q_2 = -Q$ separated by a distance d. They form an electric dipole. What are the electric field and the electric potential of this configuration?

We chose a coordinate system such that the z-axis passes through both charges and the origin is halfway between them. Thus $\vec{r}_1 = (0, 0, d/2)$ and $\vec{r}_2 = (0, 0, -d/2)$.

The electric potential at point $\vec{r}$ is found through superposition:

$$V(\vec{r}) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{r}-\vec{r}_1|} - \frac{1}{|\vec{r}-\vec{r}_2|}\right)$$

$$V(\vec{r}) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) = \frac{Q}{4\pi\epsilon_0}\left(\frac{r_- - r_+}{r_- r_+}\right)$$



From the figure we can deduce

$$r_+^2 = r_-^2 + d^2 - 2r_- d \cos\vartheta'$$

$$r_+^2 - r_-^2 = (r_+ + r_-)(r_+ - r_-) = d(d - 2r_- \cos\vartheta')$$

We get an exact solution for the potential:

$$V = \frac{1}{4\pi\epsilon_0} \frac{Qd(2r_- \cos\vartheta' - d)}{r_- r_+ (r_- + r_+)}$$

If the dipole is small compared to the distance at which we want to evaluate the potential we have $d << r$ and we can approximate $r_+ = r_- = r$ and $\vartheta' = \vartheta$.

$$V = \frac{1}{4\pi\epsilon_0} \frac{Qd \cos\vartheta}{r^2}$$

We can rewrite this using the scalar product between the vector $\vec{r}$ and the vector $\vec{d}$ pointing from the negative charge to the positive charge:

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q\vec{d}\cdot\vec{r}}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$

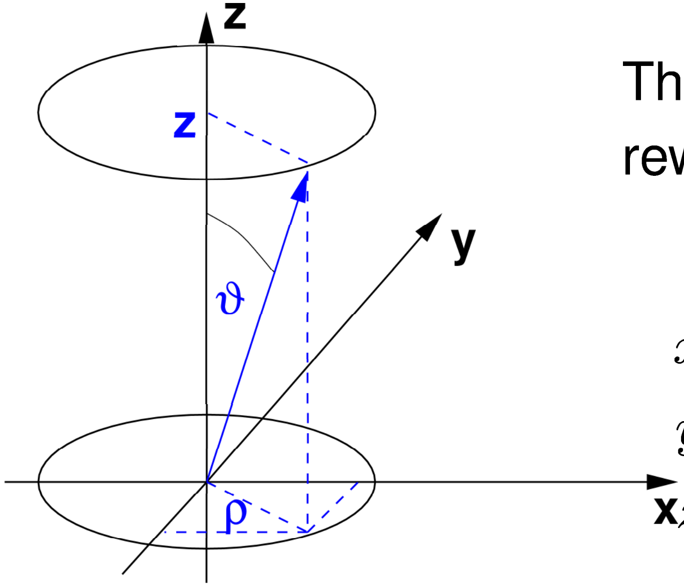
The quantity $\vec{p} = Q\vec{d}$ is the electric dipole moment.

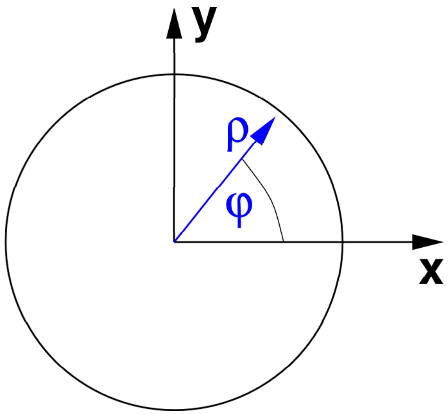
The electric field is easy: $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$. We chose $\vec{d} = (0,0,d)$ and thus we have $\vec{p} = (0,0,Qd) = (0,0,p)$:

$$\vec{E} = -\vec{\nabla}\,V = -\vec{\nabla}\,\frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} = -\vec{\nabla}\,\frac{1}{4\pi\epsilon_0}\frac{p\cdot z}{r^3}$$

We get

$$\vec{E} = \left(\frac{p}{4\pi\epsilon_0}\frac{3xz}{r^5} \quad , \quad \frac{p}{4\pi\epsilon_0}\frac{3yz}{r^5} \quad , \quad \frac{p}{4\pi\epsilon_0}\frac{3z^2-r^2}{r^5}\right)$$



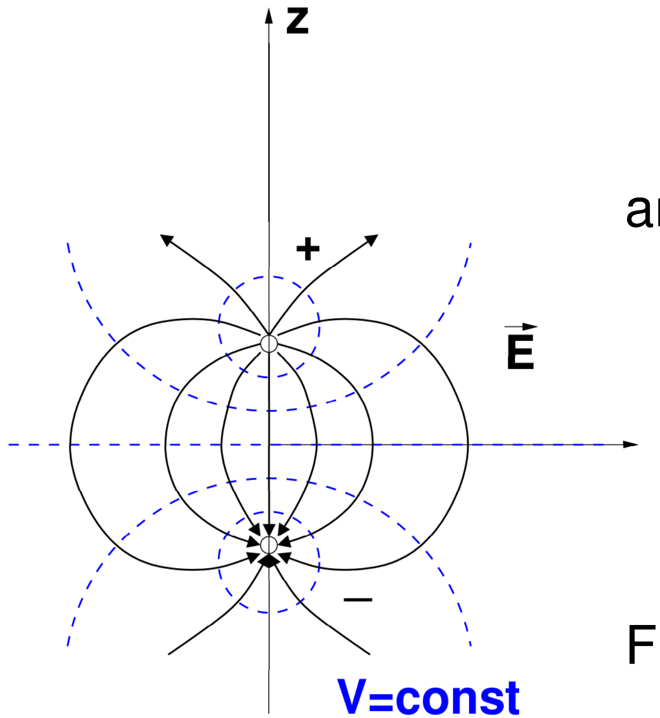


This is cumbersome in cartesian coordinates. We shall rewrite in cylindrical coordinates (ρ, ϕ, z):

$$\begin{aligned}
x &= \rho \cos\phi \\
y &= \rho \sin\phi \\
z &= z \\
r &= \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 \cos^2\phi + \rho^2 \sin^2\phi + z^2} \\
&= \sqrt{\rho^2 + z^2}
\end{aligned}$$

If we also retain the angle ϑ between the z-axis and $\vec{r}$ we can write $z = r\cos\vartheta$ and $\rho = r\sin\vartheta$

$$E_\rho = \sqrt{E_x^2 + E_y^2} = \frac{p}{4\pi\epsilon_0}\frac{3z}{r^5}\sqrt{x^2+y^2}$$
$$= \frac{p}{4\pi\epsilon_0}\frac{3z\rho}{r^5}$$
$$= \frac{p}{4\pi\epsilon_0}\frac{3\sin\vartheta\cos\vartheta}{r^3}$$



and

$$E_z = \frac{p}{4\pi\epsilon_0}\frac{3z^2 - r^2}{r^5}$$
$$= \frac{p}{4\pi\epsilon_0}\frac{(3\cos^2\vartheta - 1)}{r^3}$$

Finally

$$E_\phi = 0$$

The electric dipole field falls off as $1/r^3$

2.18: Summary

The circuital law in differential form is

$$\mathrm{curl}\vec{E}(\vec{r}) = 0 \qquad \text{or} \qquad \vec{\nabla} \times \vec{E}(\vec{r}) = 0$$

We found $\quad \mathrm{curl\ grad}\ \Phi = 0 \quad$ and $\quad \mathrm{div\ curl}\ \vec{F} = 0.$

The Poisson and Laplace equations give the electric potential from the charge distribution ρ and appropriate boundary conditions:

$$\text{Poisson:} \qquad \nabla^2 V = -\frac{\rho}{\epsilon_0} \qquad\qquad \text{Laplace:} \qquad \nabla^2 V = 0$$

The electric potential is unique

An electric dipole has a potential

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q\vec{d}\cdot\vec{r}}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$

The quantity $\vec{p} = Q\vec{d}$ is the electric dipole moment.

The electric field of a dipole along the z-axis is given in cylinder coordinates:

$$\vec{E}(\vec{r}) = \frac{p}{4\pi\epsilon_0\, r^3}\left(3\sin\vartheta\cos\vartheta\,,\;3\cos^2\vartheta - 1\,,\;0\right)$$

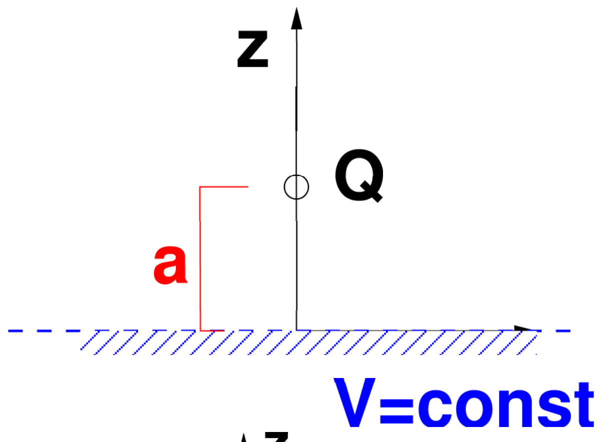
2.19: Method of Images

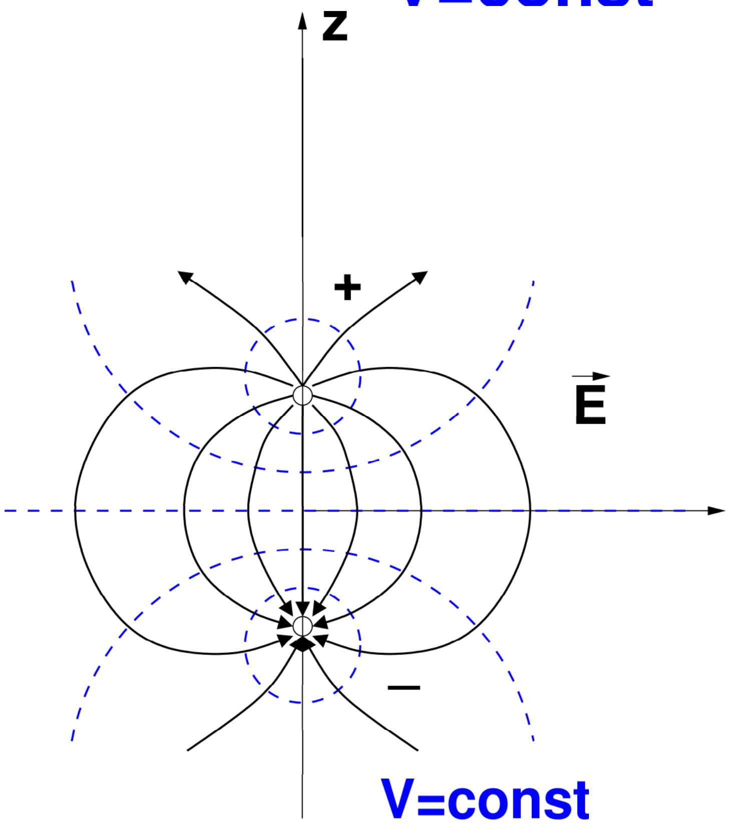
We shall use the uniqueness theorem to calculate the electric field and potential in certain highly symmetric situations.

Consider an infinite conducting plane held at potential $V = 0$ in the x-y plane.

A charge Q is located a distance a above the plane at $\vec{r}_a = (0, 0, a)$. Calculate the electric field in the top half of space $(z \geq 0)$

Gaussian surfaces won't help, because the symmetry of a point charge is broken. We could solve $\nabla^2 V = -\rho_0/\epsilon_0$ with boundary condition $V(z = 0) = 0$ but that would be tedious.





Consider now the electric dipole:

The dashed line represents an equipotential surface. Lets choose $V = 0$ there.

The problem now looks exactly like the first one for $z \geq 0$ and from last chapter we know the solutions: The electric dipole field.

$\nabla^2 V = -\rho_0/\epsilon_0$ is solved for $z \geq 0$ with the same boundary conditions, thus the uniqueness theorem says it must be the only solution.

Furthermore an image charge must be induced in the conductor. Can we calculate that?

Remember we found earlier that the electric field is normal to any equipotential surface and that the magnitude is equal to the surface charge density σ/ϵ_0:

In our case that means $\vec{E}(z=0) = (0, 0, E_z(z=0))$.

We cannot just use the solution for the field far away from the dipole because it has a pole at the origin.

$$\begin{aligned}
\vec{E} &= \sum \frac{q_i}{4\pi\epsilon_0} \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3} \\
&= \frac{-Q}{4\pi\epsilon_0} \frac{(x, y, a)}{(\sqrt{x^2+y^2+a^2})^3} + \frac{Q}{4\pi\epsilon_0} \frac{(x, y, -a)}{(\sqrt{x^2+y^2+a^2})^3} \\
E_z &= \frac{-Q}{2\pi\epsilon_0} \frac{a}{(\sqrt{x^2+y^2+a^2})^3} = \frac{-Q}{2\pi\epsilon_0} \frac{a}{(\sqrt{\rho^2+a^2})^3}
\end{aligned}$$

Which in turn gives us the charge density

$$\sigma = \frac{-Qa}{2\pi(\sqrt{\rho^2+a^2})^3}$$

Second Example:

A long thin wire is uniformly charged with a charge density λ. Its electric field can be found from Gauss's law with a cylindrical surface around the wire as

$$2\pi r\, z\, E(r) = \frac{\lambda\, z}{\epsilon_0}$$

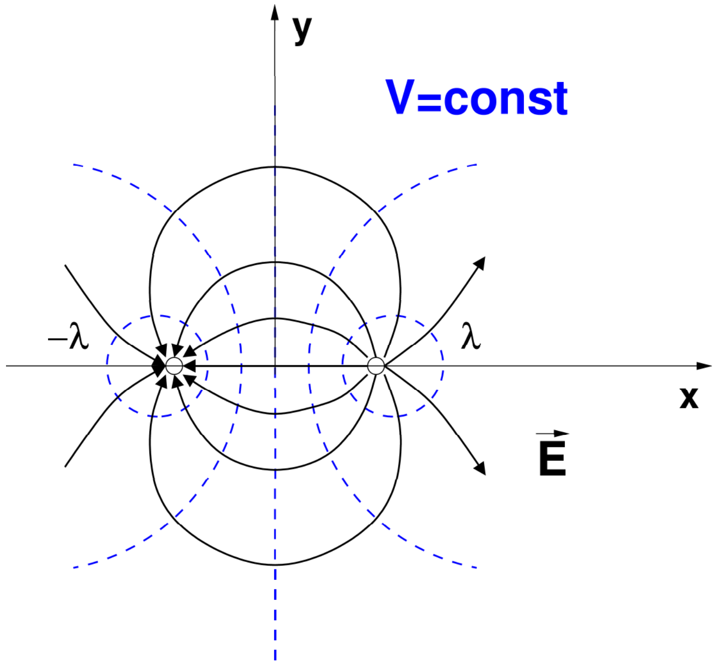
or

$$\vec{E}_r(r) = \frac{\lambda}{2\pi\epsilon_0 r}\,\hat{\underline{\mathsf{a}}}_r$$

and

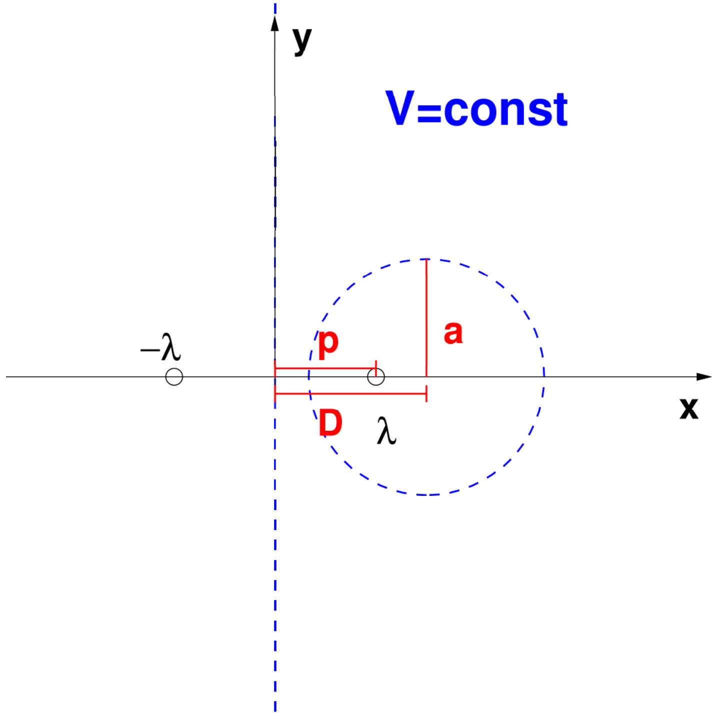
$$V(r) = \frac{-\lambda}{2\pi\epsilon_0}\,\ln r$$

if we choose $V = 0$ at infinity.

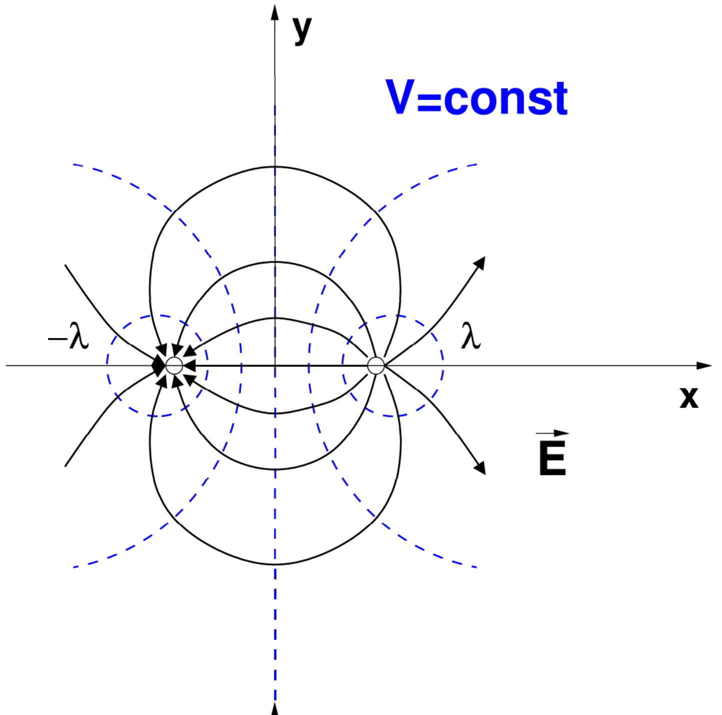


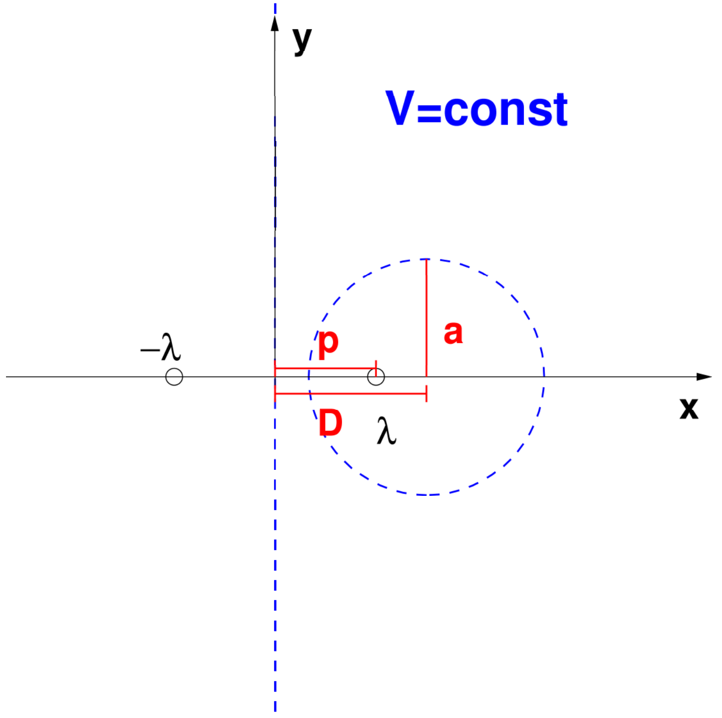
We now take two wires parallel to the z-axis crossing the x-y plane at $\vec{r}_1 = (+p, 0, 0)$ and $\vec{r}_2 = (-p, 0, 0)$ with charge densities $+\lambda$ and $-\lambda$ respectively.

The potentials add to give $V(\vec{r}) = V_1(\vec{r}) + V_2(\vec{r})$



$$
\begin{aligned}
V(\vec{r}) &= \frac{-\lambda \ln(|\vec{r}-\vec{r}_1|)}{2\pi\epsilon_0} + \frac{\lambda \ln(|\vec{r}-\vec{r}_2|)}{2\pi\epsilon_0} \\
&= \frac{\lambda}{2\pi\epsilon_0} \ln \frac{|\vec{r}-\vec{r}_2|}{|\vec{r}-\vec{r}_1|}
\end{aligned}
$$





First the trivial solution:

$$\frac{2\pi\epsilon_0}{\lambda} V = \text{const} \;=\; \ln \frac{\sqrt{(x+p)^2 + y^2}}{\sqrt{(x-p)^2 + y^2}}$$

$$= \frac{1}{2} \ln \frac{(x+p)^2 + y^2}{(x-p)^2 + y^2}$$

$$\rightarrow (x+p)^2 + y^2 = (x-p)^2 + y^2$$

$$4xp = 0$$

$$\rightarrow x = 0$$

An equipotential plane in the y-z plane.

We will now show that the equipotential surfaces are circular cylinders with radius a centered on a point at distance $D = \sqrt{p^2 + a^2}$ from the origin:

$$\frac{2\pi\epsilon_0}{\lambda} V = \text{const} = \frac{1}{2} \ln \frac{(x+p)^2 + y^2}{(x-p)^2 + y^2}$$

A cylinder of radius a around $\vec{D} = (\sqrt{a^2 + p^2}, 0, 0)$ is given by $a^2 = (x - D)^2 + y^2$ or $y^2 = a^2 - (x - D)^2$.

The argument of the ln is:

$$\frac{(x+p)^2 + a^2 - (x-D)^2}{(x-p)^2 + a^2 - (x-D)^2} = \frac{x^2 + 2xp + p^2 + a^2 - x^2 + 2xD - D^2}{x^2 - 2xp + p^2 + a^2 - x^2 + 2xD - D^2}$$

But $D^2 = p^2 + a^2$

This leaves

$$\frac{2x(D+p)}{2x(D-p)} = \frac{D+p}{D-p} = \text{const}$$

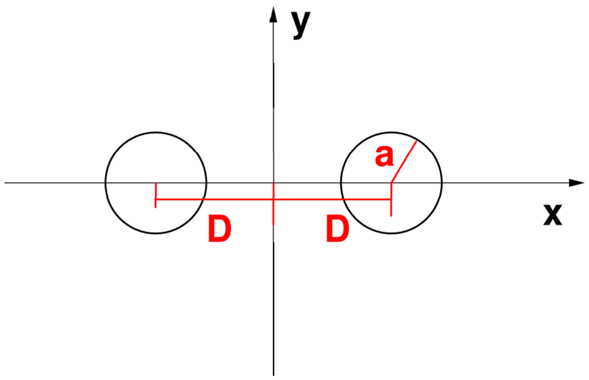
and

$$V = \frac{\lambda}{2\pi\epsilon_0} \ln\sqrt{\frac{D+p}{D-p}} = \frac{\lambda}{4\pi\epsilon_0} \ln\frac{D+p}{D-p} = \text{const on these cylinders.}$$

The electric field follows.

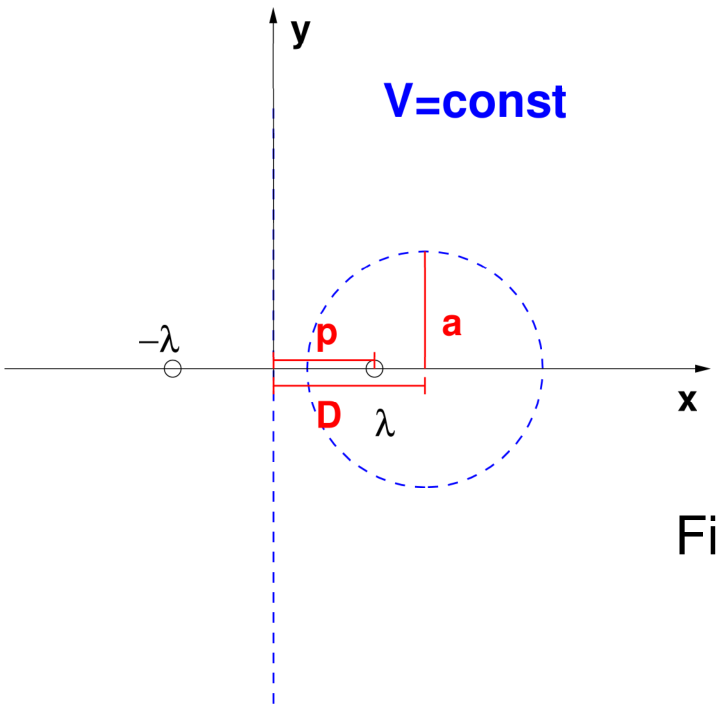
This can be used to solve electrostatic problems involving planes and cylinders, e.g.: What is the capacitence per unit length between two long circular wires of radius a with equal and opposite charges $\pm\lambda$?

We can choose image charges as before through $x = \pm p$ with $p^2 + a^2 = D^2$ and have exactly the previous situation. The potential on the surface of each wire is



$$V_\pm = \frac{\pm\lambda}{4\pi\epsilon_0} \ln \frac{D+p}{D-p}$$

$$\begin{aligned}
\rightarrow \Delta V = V_+ - V_- &= \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{D+p}{D-p}\right) \\
&= \frac{\lambda}{2\pi\epsilon_0} \ln \frac{(D+p)^2}{D^2-p^2} \\
&= \frac{\lambda}{2\pi\epsilon_0} \ln \frac{(D+p)^2}{a^2} \\
&= \frac{\lambda}{\pi\epsilon_0} \ln \frac{(D+p)}{a}
\end{aligned}$$



Finally we get for the capacitance per unit length:

$$C = \frac{\lambda}{\Delta V} = \frac{\pi\epsilon_0}{\ln\left(\frac{D+p}{a}\right)} \simeq \frac{\pi\epsilon_0}{\ln\left(\frac{2D}{a}\right)}$$

2.20: Gauss' and Stokes' Theorems

We remember without proof two general theorems:

Gauss' theorem (also called divergence theorem):

For any closed surface S bounding a volume V and a vector field F defined everywhere in V and on S which is continuous and differentiable we have

$$\oint_S \vec{F} \, \underline{\mathrm{d}\, S} = \int_V \mathrm{div}\vec{F} \, \mathrm{d}\, V$$

Stoke's theorem:

For any oriented surface S bounded by a closed curve C and a vector field $\vec{F}$ as above we have

$$\oint_C \vec{F} \, \underline{\mathrm{d}\, l} = \int_S \mathrm{curl}\vec{F} \, \underline{\mathrm{d}\, S}$$

Compare with 1 dimension $\qquad \displaystyle\int_a^b \frac{\mathrm{d}\, f}{\mathrm{d}\, x} \mathrm{d}\, x = f(b) - f(a)$

2.21: Summary

In certain cases we can use the method of images to find the solutions to the Laplace equation. The uniqueness theorem ensures that any solution found is unique if the boundary conditions are sufficiently well defined.

The equipotential surfaces around straight wires are cylinders, the equipotential surfaces around point charges are spheres.

Gauss' theorem (also called divergence theorem):

$$\oint_S \vec{F}\, \underline{\mathrm{d}\, S} = \int_V \mathrm{div}\vec{F}\, \mathrm{d}\, V$$

Stoke's theorem:

$$\oint_C \vec{F}\, \underline{\mathrm{d}\, l} = \int_S \mathrm{curl}\vec{F}\, \underline{\mathrm{d}\, S}$$

2.22: Steady electric currents: The concept of electromotance

A steady current consists of charges moving at uniform velocity. We describe it through the current density $\vec{j} = nq\vec{v} = -ne\vec{v}$

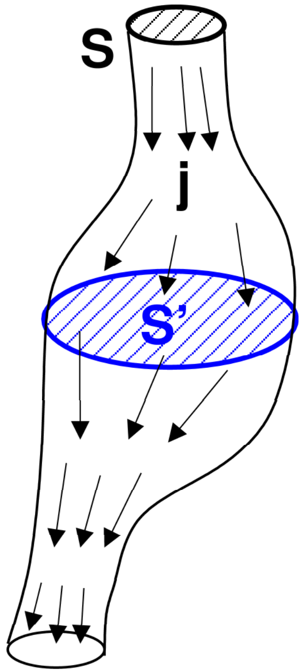
n: density of charge carriers (conductive electrons)
$-e$: charge of an electron (charge carrier in most materials)
$\vec{v}$: velocity

The current I through any surface S is the net charge that passes through the surface per second:

$$I = \int_S \vec{j} \, \mathrm{d}\, \underline{S}$$

Units:
Current: charge/time = Coulomb / s = Ampere
Current density: current/area = A m^{-2}
Note: $\vec{j}$ is a vector, I is not!

Take any conductor maintaining a steady current



$$\int_S \vec{j}\, \underline{\mathrm{d}\, S} = \int_{S'} \vec{j}'\, \underline{\mathrm{d}\, S'}$$

In most materials the charges move in the direction of the local electric field:

$\vec{j} = \sigma \vec{E}$ σ is the conductivity (in $1/(\Omega m)$)

This is the differential from of Ohm's Law.

$$I = \int_S \vec{j}\, \underline{\mathrm{d}\, S} = \sigma \int_S \vec{E} \underline{\mathrm{d}\, S}$$

Materials may be oriented, and the conductivity may be a function of the crystal structure. In general $\underline{\underline{\sigma}}$ will be a tensor (i.e. a 3x3 matrix) that takes any such material properties into account:

$$\vec{j} = \underline{\underline{\sigma}}\vec{E}$$

In this course we will not concern ourselves with such media.

If the electric field is generated by a constant potential difference (e.g. a battery) in a uniform conductor then $|E| = V/l$ and we get

$$I = \sigma \frac{V}{l} \int_S \mathrm{d}\, S = \sigma \frac{V\,A}{l}$$

The resistance is then

$$R = \frac{V}{I} = \frac{l}{\sigma A} = \rho \frac{l}{A}$$

The resistivity ρ and conductivity σ are directly related: $\rho = 1/\sigma$.

As the current density deals with moving charges and charge is conserved we can derive a conservation law:

Consider a volume V bounded by a closed surface S containing charges $\rho(\vec{r},t)$ moving at velocities $v(\vec{r},t)$

The total charge inside V at time t allows us to calculate the net current out of the volume:

$$\begin{aligned} Q(t) &= \int_V \rho(\vec{r},t)\,\mathrm{d}\,V \\ \frac{\mathrm{d}}{\mathrm{d}\,t}Q(t) &= \int_V \frac{\mathrm{d}}{\mathrm{d}\,t}\rho(\vec{r},t)\,\mathrm{d}\,V \end{aligned}$$

The current density is $\vec{j} = \rho(\vec{r},t)\cdot\vec{v}(\vec{r},t)$

and the net flow out of the volume is $I = \oint_S \vec{j}\,\underline{\mathrm{d}\,S}$

We use Gauss' theorem to rewrite

$$I = \oint_S \vec{j}\,\underline{\mathrm{d}\,S} = \int_V \mathrm{div}\vec{j}\,\mathrm{d}\,V = -\frac{\mathrm{d}}{\mathrm{d}\,t}\,Q(t)$$

$$\rightarrow \int_V \mathrm{div}\vec{j}\,\mathrm{d}\,V = -\int_V \frac{\mathrm{d}}{\mathrm{d}\,t}\,\rho\,\mathrm{d}\,V$$

or

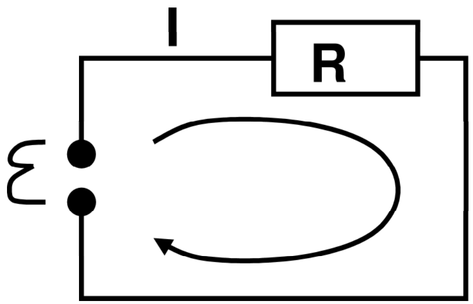
$$\mathrm{div}\vec{j} = -\frac{\mathrm{d}\,\rho}{\mathrm{d}\,t}$$

For steady currents any charge lost gets replentished

$$\rightarrow \frac{\mathrm{d}}{\mathrm{d}\,t}\rho = 0 \quad \rightarrow \quad \mathrm{div}\vec{j} = 0 \qquad \text{or} \quad \oint_S \vec{j}\underline{\mathrm{d}\,S} = 0$$

How do we maintain a steady current?

We need a source of eletromotance (EMF).



In this course we will not concern ourselves with the internal working of generators or batteries but will adopt a general source of electromotance. It has two terminals A and B and maintains both at constant potentials V_A and V_B.

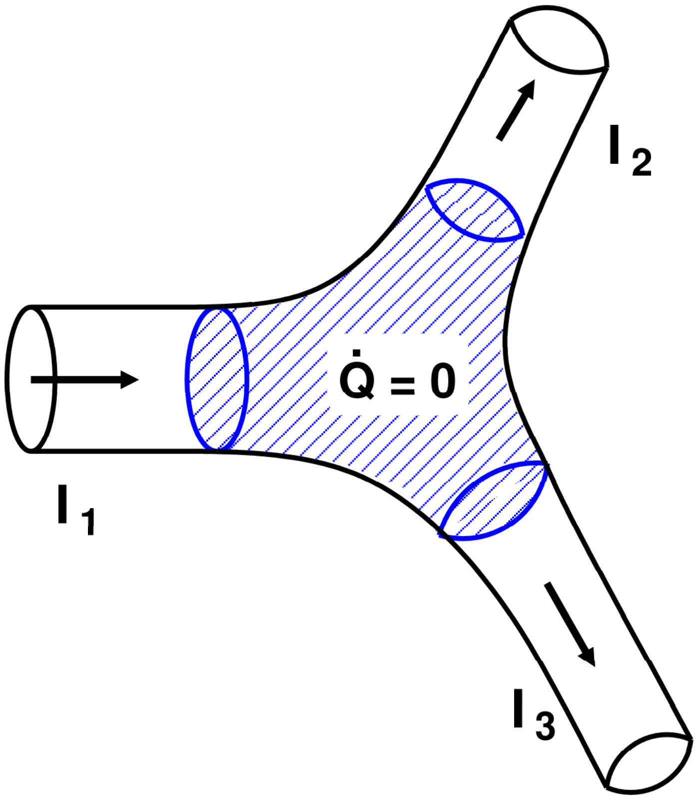
Then $\quad \mathcal{E} = V_B - V_A$

Consider now a closed circuit.

We have $\quad \mathcal{E} = I\,R$

Or in general: $\quad \Sigma\,\mathcal{E}_i = \Sigma\,I_i\,R_i$ round a closed loop.

This is Kirchhoff's second law.



Kirchhoff's first law is:

At any junction we have $\quad \Sigma I_i = 0.$

Consider the shaded volume. The net flow of charges out of this volume is $I_1 + I_2 + I_3$. However, the situation was steady, so the total charge in the shaded volume has to be constant.

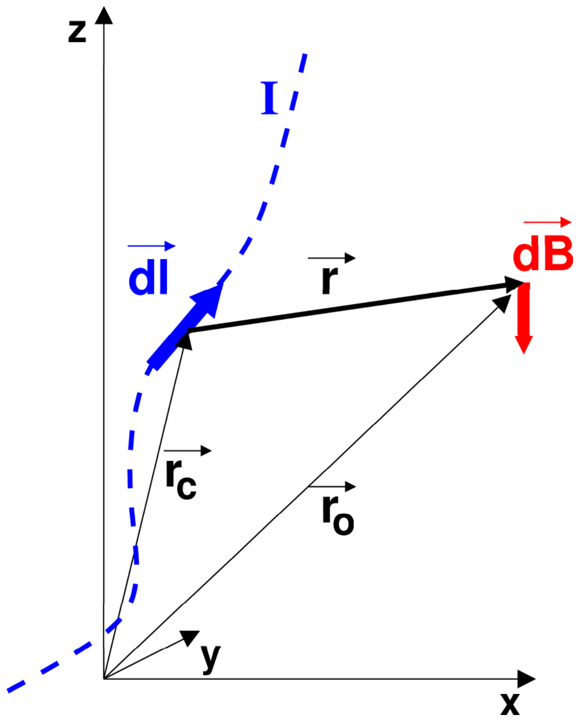
$$\rightarrow \Sigma I_i = 0$$

3: Magnetostatics in vacuum

Experimentally we know of magnetic effects and fields mainly through the forces on other objects. However, we already saw in our discussion of electrostatics that the definition of a field via a force can be problematic. Although this approach is still sometimes used today to maintain the historic perspective, we shall not burden ourselves with it and choose the magnetic field itself as the starting point of our discussion.

What is the electrostatic analogue of a charge? There are no magnetic monopoles, and to start from the dipole makes the whole discussion unneccessarily complex. We choose the Biot-Savart law as a starting point which establishes the current element as a “source” of magnetic field. It is well rooted in experimental data and we shall see later that it indeed derives from Coulomb’s law and special relativity.

3.1: The Biot-Savart law



The Biot Savart law allows us to calculate the contribution $d\vec{B}$ to the magnetic field $\vec{B}$ from a current element Idl:

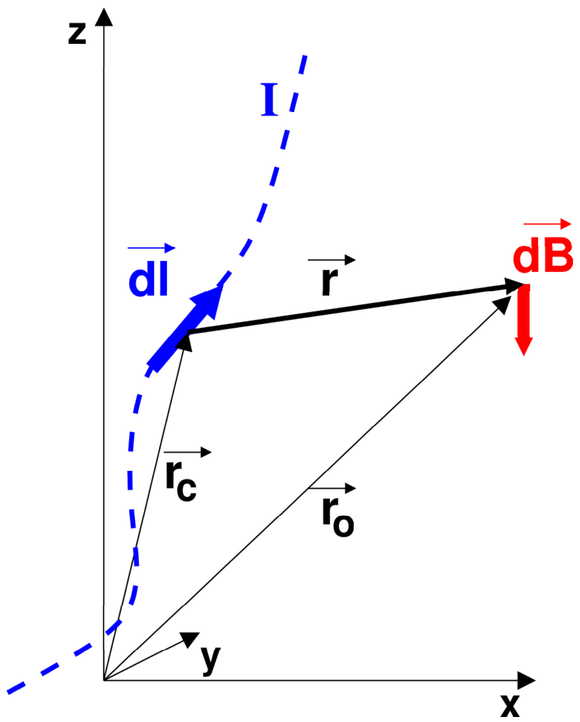
$$d\vec{B} = \frac{\mu_0 I}{4\pi r^3}\,\vec{dl} \times \vec{r}$$

The current element $\vec{dl}$ is located at $\vec{r}_c$, the magnetic field is evaluated at $\vec{r}_0$, and the vector $\vec{r} = \vec{r}_0 - \vec{r}_c$ points from the current element to the place where we evaluate $d\vec{B}$.

The proportionality constant μ_0 depends on the choice of units. In SI units we have:

$\mu_0 = 4\pi \times 10^{-7}$ H/m [= Ns^2/C^2 =kg m/C^2 =Am/(Vs) =Vs^2/(Cm) = m/Ωs =...]

In practice we will always integrate over macroscopic currents, i.e. entire circuits. Thus we get



$$\vec{B}(\vec{r}_0) = \int_C \frac{\mu_0 I}{4\pi(|\vec{r}_0 - \vec{r}_c|)^3} \vec{dl} \times (\vec{r}_0 - \vec{r}_c)$$

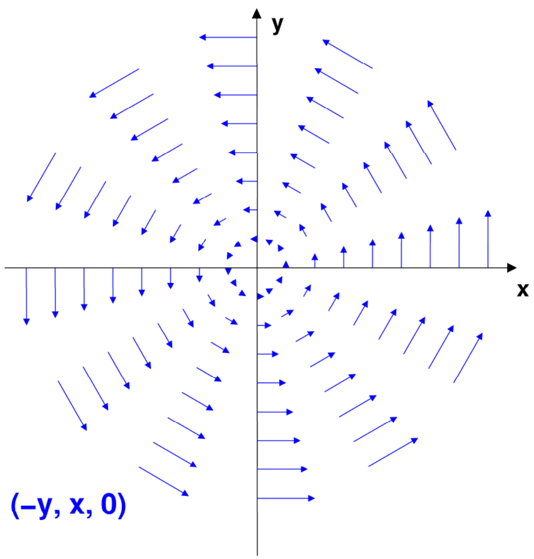
If the current is extended over a volume we can rewrite the Biot-Savart law in the following way:

$$\vec{B}(\vec{r}_0) = \int_V \frac{\mu_0}{4\pi(|\vec{r}_0 - \vec{r}_c|)^3} \vec{j} \times (\vec{r}_0 - \vec{r}_c)\, dV$$

Note the cross product. If the current element stems from a wire then $\vec{dl}$, $\vec{r}$ and $d\vec{B}$ will form a righthanded system.

3.2: Example: straight wire

We calculate the magnetic field at a point $\vec{r}_0 = (x, y, 0)$ caused by a straight wire running from a to b along the z-axis carrying a steady current I.
The current element $I\vec{dl} = I\,dz\,\hat{\underline{\mathrm{a}}}_z$ at $\vec{r}_c = (0, 0, z)$ corresponds to a current flowing in positive z-direction.



$$\vec{B}(\vec{r}_0) = \int d\vec{B} = \int_a^b \frac{\mu_0 I}{4\pi} \frac{\hat{\underline{\mathrm{a}}}_z \times (\vec{r}_0 - \vec{r}_c)}{(|\vec{r}_0 - \vec{r}_c|)^3}\, dz$$

$$= \int_a^b \frac{\mu_0 I}{4\pi} \frac{(-y, x, 0)}{\left(\sqrt{x^2 + y^2 + z^2}\right)^3}\, dz$$

We have done such integrals before and found:

$$\int_a^b \frac{dx}{\left(\sqrt{x^2+k^2}\right)^3} = \left[\frac{x}{k^2\sqrt{x^2+k^2}}\right]_a^b$$

Thus we get for the components of $\vec{B}$:

$$\begin{aligned}
B_x &= \int_a^b \frac{\mu_0 I}{4\pi}\frac{-y\,dz}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{-y\mu_0 I}{4\pi(x^2+y^2)}\left(\frac{b}{\sqrt{x^2+y^2+b^2}} - \frac{a}{\sqrt{x^2+y^2+a^2}}\right)\\
B_y &= \int_a^b \frac{\mu_0 I}{4\pi}\frac{x\,dz}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{x\mu_0 I}{4\pi(x^2+y^2)}\left(\frac{b}{\sqrt{x^2+y^2+b^2}} - \frac{a}{\sqrt{x^2+y^2+a^2}}\right)\\
B_z &= 0
\end{aligned}$$

$$\vec{B}(\vec{r}_0) = \frac{\mu_0 I}{4\pi(x^2+y^2)}\left(\frac{b}{\sqrt{x^2+y^2+b^2}} - \frac{a}{\sqrt{x^2+y^2+a^2}}\right)(-y,x,0) = B_0(-y,x,0)$$

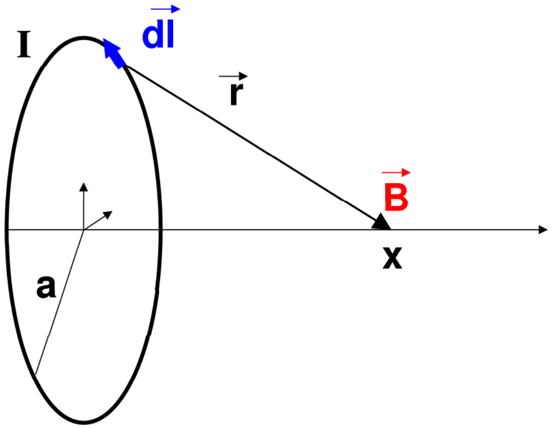
For an infinite wire we take $a = -b$ and let b tend to infinity:

$$
\begin{aligned}
\vec{B}(\vec{r}_0) &= \lim_{b\to\infty} \frac{\mu_0 I}{4\pi(x^2+y^2)} \left(\frac{b}{\sqrt{x^2+y^2+b^2}} - \frac{-b}{\sqrt{x^2+y^2+b^2}} \right) (-y, x, 0) \\
&= \frac{\mu_0 I}{2\pi(x^2+y^2)} (-y, x, 0) \\
|\vec{B}| &= \frac{\mu_0 I}{2\pi\sqrt{x^2+y^2}} = \frac{\mu_0 I}{2\pi\rho}
\end{aligned}
$$

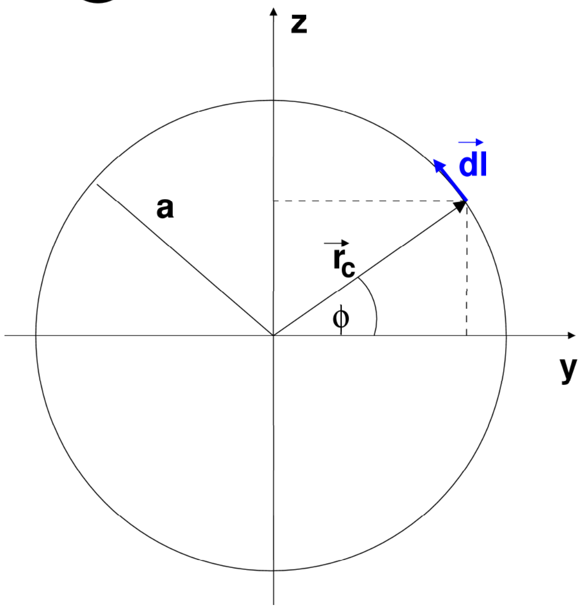
The last form again makes use of cylinder coordinates.

The field lines do form right handed circles around the wire.

3.3: Example 2: Field on the axis of a circular coil



Here we shall consider a circular loop of radius a in the y-z plane. The loop carries a current I moving clockwise looking in positive x-direction and we want to evaluate the magnetic field on a point $\vec{r}_0 = (x, 0, 0)$ on the x-axis.



We can parametrize the loop with the angle ϕ. Then the line element $\vec{dl}$ at $\vec{r}_c = (0, a\cos\phi, a\sin\phi)$ is

$$\vec{dl} = a\, d\phi\, (0, -\sin\phi, \cos\phi)$$

and

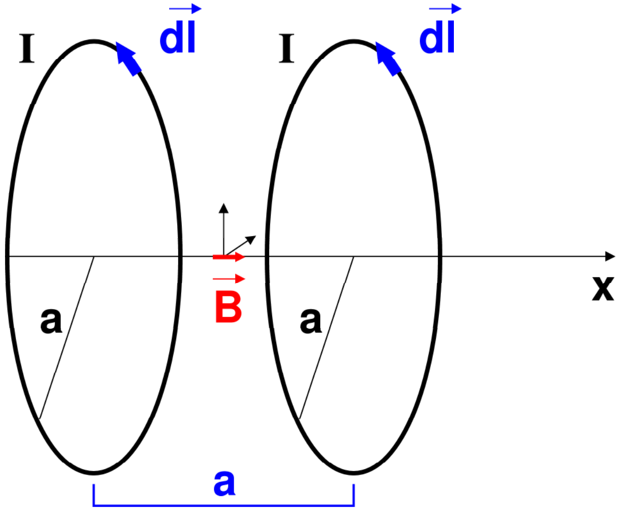
$$\begin{aligned}\vec{dl} \times \vec{r} &= a\, d\phi\, (0, -\sin\phi, \cos\phi) \times (x, -a\cos\phi, -a\sin\phi) \\ &= (a^2, xa\cos\phi, xa\sin\phi)\, d\phi\end{aligned}$$

The magnetic field then becomes

$$\vec{B}(\vec{r}_0) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{(a^2, ax\cos\phi, ax\sin\phi)}{\left(\sqrt{x^2+a^2}\right)^3}\,d\phi$$

$$= \frac{\mu_0 I a^2}{2\left(\sqrt{x^2+a^2}\right)^3}\underline{\hat{\mathsf{a}}}_x$$

An application are Helmholtz coils. In experiments one often wants a homogeneous magnetic field in a certain region of space while retaining easy access to any experimental apparatus placed in this region. Here Helmholtz coils are important: Two identical current loops with radius a are placed a distance a apart. The magnetic field on the axis is very homogeneous over a large region of space.



Using the results we just obtained we get for the magnetic field

$$\vec{B}(\vec{r}=(x,0,0)) = \frac{\mu_0}{2}\left[\frac{a^2}{\left(\sqrt{(x-a/2)^2+a^2}\right)^3} + \frac{a^2}{\left(\sqrt{(x+a/2)^2+a^2}\right)^3}\right]\underline{\hat{a}}_x$$

In the center between the coils we get the field:

$$\vec{B}(\vec{r}=\vec{0}) = \frac{\mu_0}{2}\left[\frac{a^2}{\left(\sqrt{a^2/4+a^2}\right)^3} + \frac{a^2}{\left(\sqrt{a^2/4+a^2}\right)^3}\right]\underline{\hat{a}}_x$$

$$= \frac{\mu_0 I a^2}{a^3\sqrt{5/4^3}}\underline{\hat{a}}_x = \frac{8\mu_0 I}{a\sqrt{125}}\underline{\hat{a}}_x$$

3.4: Summary

The current I through any surface S is the net charge that passes through the surface per second:

$$I = \int_S \vec{j}\, \underline{\mathrm{d}\, S}$$

The differential from of Ohm's Law:

$$\vec{j} = \sigma\, \vec{E} \qquad\qquad \sigma \text{ is the conductivity (in } 1/(\Omega m))$$

Conservation of charge:

$$\mathrm{div}\vec{j} = -\frac{\mathrm{d}\,\rho}{\mathrm{d}\,t}$$

Kirchhoffs laws in circuits are:

1^{st} law:

At any junction we have $\Sigma I_i = 0$.

2^{nd} law:

$\Sigma\,\mathcal{E}_i = \Sigma\, I_i\, R_i$ around any closed loop.

The Biot Savart law allows us to calculate the contribution $d\vec{B}$ to the magnetic field $\vec{B}$ from a current element I<u>dl</u>:

$$d\vec{B} = \frac{\mu_0 I}{4\pi r^3}\, \vec{dl} \times \vec{r}$$

$\mu_0 = 4\pi \times 10^{-7}$ H/m [= Ns^2/C^2 =kg m/C^2 =Am/(Vs) =Vs^2/(Cm) = m/Ωs =...]

In practice we will always integrate over macroscopic currents, i.e. entire circuits. Thus we get

$$\vec{B}(\vec{r}_0) = \int_C \frac{\mu_0 I}{4\pi(|\vec{r}_0 - \vec{r}_c|)^3}\,\vec{dl} \times (\vec{r}_0 - \vec{r}_c)$$

If the current is extended over a volume we can rewrite the Biot-Savart law in the following way:

$$\vec{B}(\vec{r}_0) = \int_V \frac{\mu_0}{4\pi(|\vec{r}_0 - \vec{r}_c|)^3}\,\vec{j} \times (\vec{r}_0 - \vec{r}_c)\,dV$$

3.5: Forces between Magnetic Field and Current

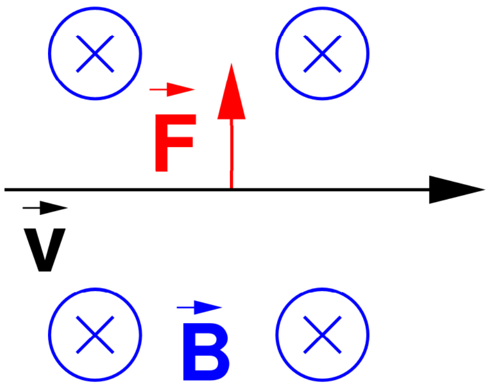
The force on a charge Q_i in an electric field $\vec{E}$ was $\vec{F} = Q \cdot \vec{E}$

What is the force on a charge from a magnetic field $\vec{B}$?

$$\vec{F} = Q\,\vec{v} \times \vec{B}$$

This form is supported by experiment and has been shown to be correct to great accuracy. We can derive it formally from the Coulomb force using special relativity.

In the presence of both electric and magnetic fields the total force is the Lorentz force:



$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$$

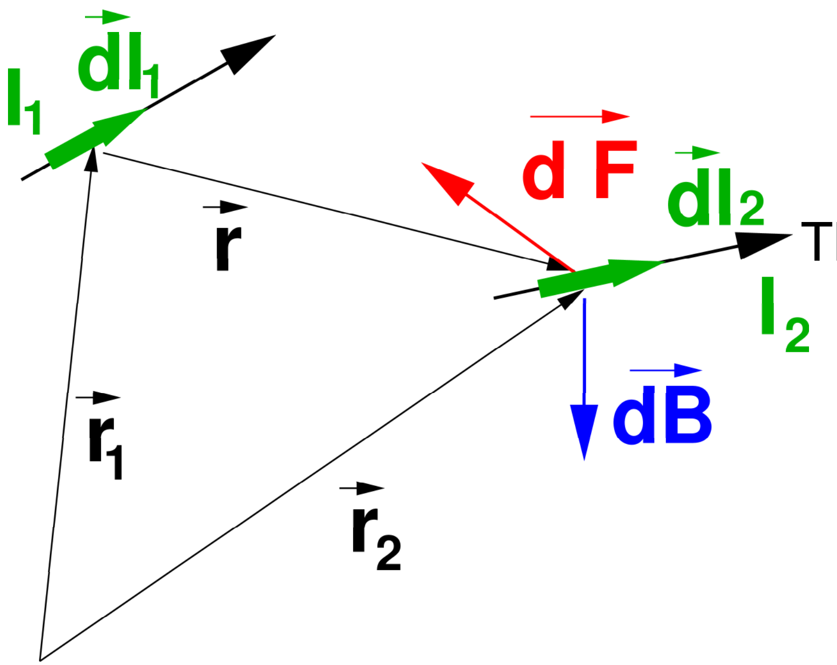
This allows us to investigate the forces on currents in magnetic fields:

$$\vec{F} = \int_V \rho\,\vec{v} \times \vec{B}\,\mathrm{d}\,V = \int_V \vec{j} \times \vec{B}\,\mathrm{d}\,V$$

or, if we take a current through a wire $\vec{F} = \int_L I\,\mathrm{d}\,\vec{L} \times \vec{B}$

3.6: Force between Currents

The Biot-Savart law gave us a magnetic field as a consequence of currents. The Lorentz force gives a force as a consequence of the presence of a current in a magnetic field. So can we derive the force between two current elements?



First, call the magnetic field at $\vec{r}_2$ due to $I_1\,\vec{dl}_1$

$$\mathrm{d}\,\vec{B} = \frac{\mu_0}{4\,\pi}\frac{I_1\,\vec{dl}_1 \times (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$

The Lorentz force between I_2 and $d\vec{B}$ is

$$\mathrm{d}^{\,2}\vec{F}_L = I_2 d\vec{I}_2 \times d\vec{B}$$

$$\mathrm{d}^{\,2}\vec{F}_L = \frac{\mu_0 I_1 I_2}{4\,\pi}\frac{\vec{dl}_2 \times (\vec{dl}_1 \times (\vec{r}_2 - \vec{r}_1))}{|\vec{r}_2 - \vec{r}_1|^3}$$

This is a second order differential – to get the force between two finite circuits we have to integrate over both $\vec{dl}_1$ and $\vec{dl}_2$

$$\rightarrow \vec{F}_{12} = \oint_{L_1} \oint_{L_2} \frac{\mu_0 I_1 I_2}{4\pi} \frac{\vec{dl}_2 \times (\vec{dl}_1 \times (\vec{r}_2 - \vec{r}_1))}{|\vec{r}_2 - \vec{r}_1|^3}$$

If we denote the force element on $I_2 \vec{dl}_2$ caused by $I_1 dl_1$ by $\mathrm{d}^2 \vec{F}_{12}$ then we can see that

$$\mathrm{d}^2 \vec{F}_{12} \neq -\mathrm{d}^2 \vec{F}_{21}$$

However, if integrated over the whole circuits we get Newton's law back:

$$\vec{F}_{12} = -\vec{F}_{21}$$

$$\begin{aligned}\vec{F}_{12} &= \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{\vec{dl}_2 \times (\vec{dl}_1 \times (\vec{r}_2 - \vec{r}_1))}{|\vec{r}_2 - \vec{r}_1|^3} \\ &= \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{\vec{dl}_1(\vec{dl}_2 \cdot (\vec{r}_2 - \vec{r}_1)) - (\vec{r}_2 - \vec{r}_1)(\vec{dl}_1 \cdot \vec{dl}_2)}{|\vec{r}_2 - \vec{r}_1|^3}\end{aligned}$$

The first part of the integrand gives no contribution:

$$\oint_{L_2} \vec{dl}_1 \frac{\vec{dl}_2 \cdot (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3} = \vec{dl}_1 \oint_{L_2} \frac{\vec{dl}_2 \cdot (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$

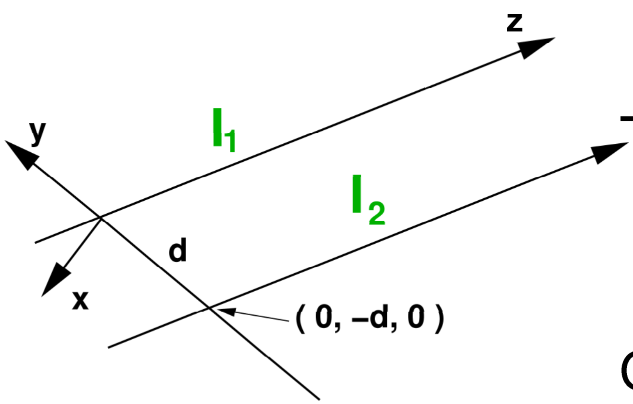
This looks like a closed loop in the field of a static electric charge. But we have shown that the integral around any closed loop in an electrostatic field vanishes. We are left with

$$\vec{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{-(\vec{r}_2 - \vec{r}_1)(\vec{dl}_1 \cdot \vec{dl}_2)}{|\vec{r}_2 - \vec{r}_1|^3}$$

This is symmetric.

We can now calculate the force per unit length between two infinite straight wires separated by a distance d carrying currents I_1 and I_2. We choose one wire to lie in the z-axis and the other to pass through the point $(0, -d, 0)$. We already calculated the field $\vec{B}$ caused by an infinite straight wire lying on the z-axis to be

$$\vec{B}(\vec{r}) = \frac{I_1 \mu_0}{2\pi(x^2 + y^2)}(-y, x, 0)$$



This gives the magnetic field at the position of the other wire as

$$\vec{B}(\vec{r} = (0, -d, z)) = \frac{I_1 \mu_0}{2\pi(d^2)}(d, 0, 0) = \frac{I_1 \mu_0}{2\pi d}\underline{\hat{\mathrm{a}}}_x$$

The Lorentz force on a piece of wire $\vec{dl}_2 = dl_2 \underline{\hat{\mathrm{a}}}_z$ becomes

$$d\vec{F}_L = I_2 \vec{dl}_2 \times \vec{B} = \frac{I_1 I_2 \mu_0}{2\pi d} dz(\underline{\hat{\mathrm{a}}}_z \times \underline{\hat{\mathrm{a}}}_x) = \frac{I_1 I_2 \mu_0}{2\pi d} dz \underline{\hat{\mathrm{a}}}_y$$

Or integrated over a length l:

$$\vec{F}_L = \int_0^l d\vec{F}_L = \frac{I_1 I_2 \mu_0 l}{2\pi d}\underline{\hat{\mathrm{a}}}_y$$

towards the first wire! Two parallel currents attract each other.

If we choose $I_1 = I_2 = 1\,\mathrm{A}$, $d = 1\,\mathrm{m}$, $l = 1\,\mathrm{m}$ we get

$$|\vec{F}| = \frac{1\,\mathrm{A} \cdot 1\,\mathrm{A} \cdot 1\,\mathrm{m} \cdot 4\pi \times 10^{-7}\mathrm{N/A}^2}{2\pi \cdot 1\,\mathrm{m}} = 2 \times 10^{-7}\,\mathrm{N}$$

This is the SI definition of the Ampere:

“One Ampere is that current which, if maintained in two straight parallel conductors of infinite length, neglegible cross section, placed 1 m apart in vacuum, would produce between these conductors a force equal to $2 \times 10^{-7}\,\mathrm{N}$ per metre of length.”

SI units so far:

Current	I	A							
Charge	Q	C	=	A s					
Force	$\vec{F}$	N	=	kg m/s^2					
electrostatic Potential	V	V	=	J/C	=	kg m^2/(A s^3)			
electric Field	$\vec{E}$	V/m	=	N/C	=	kg m/(A s^3)			
magnetic Field	$\vec{B}$	T	=	Wb/m^2	=	N/(A m)	=	kg/(A s^2)	
resistivity	ρ	Ω m	=	V m/A					
permittivity of free space	ϵ_0	8.854×10^{-12} As/(V m)	=	A^2 s^4/(kg m^3)					
permeability of free space	μ_0	$4\pi \times 10^{-7}$ Vs/(A m)	=	kg m/(A^2 s^2)	=	Tm/A			

3.7: Examples

Consider a charge q moving with a velocity $\vec{v}$ in a homogeneous magnetic field: $\vec{B}$:

$$\vec{B} = B_0 \underline{\hat{\mathsf{a}}}_z \qquad\qquad \vec{v} = (v_x, v_y, v_z)$$

The Lorentz force is

$$\begin{aligned} \vec{F} &= q\vec{v} \times \vec{B} = qB_0(v_y, -v_x, 0) \\ \vec{F} &= m(\ddot{x}, \ddot{y}, \ddot{z}) = qB_0(v_y, -v_x, 0) \end{aligned}$$

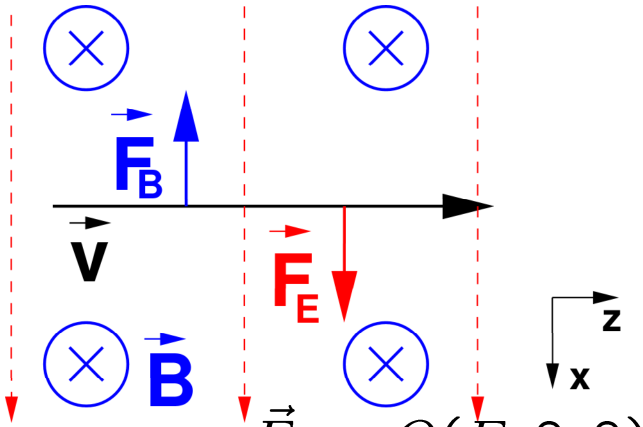
No force acts parallel to $\vec{B}$, we have uniform steady motion in z-direction. In the x-y plane we get

$$\begin{aligned} m\ddot{x} &= qB_0\dot{y} \\ m\ddot{y} &= -qB_0\dot{x} \end{aligned}$$

It is easy to show that $x(t) = A\sin(\omega t + \phi_0) + x_0$ and $y(t) = A\cos(\omega t + \phi_0) + y_0$ are solutions of this system of equations. The initial conditions determine the constants: $\omega = qB_0/m$, $A = m\sqrt{v_x^2 + v_y^2}/(qB_0) = mv_\perp/(qB_0)$

These are circles in the x-y plane. Together with the uniform motion in z-direction the most general motion of a charge q in a uniform magnetic field $\vec{B}$ is a helix around the magnetic field direction.

3.8: Velocity Filter



The Lorentz force is $\vec{F}_L = Q(\vec{E} + \vec{v} \times \vec{B})$.
If we choose $\vec{E} = E\underline{\hat{a}}_x$, $\vec{v} = v\underline{\hat{a}}_z$, and $\vec{B} = B\underline{\hat{a}}_y$ we get

$$\vec{F}_L = Q(E, 0, 0) + Q(-vB, 0, 0) = Q(E - vB)\underline{\hat{a}}_x$$

The Lorentz force vanishes for $v = E/B$. This configuration is called a Wien filter. It is extremely useful in experiments with charged particles.

A heavy ion with a velocity v enters a Wien filter with a magnetic field $B = 0.2$ T and an electric field with 200 000 V/m.

$$E/B = 10^6 \frac{\text{V}}{\text{T m}} = 10^6 \text{ m/s}$$

3.9: Summary

In the presence of both electric and magnetic fields the total force on a moving charge is the Lorentz force:

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$$

The force between two currents is

$$\vec{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{-(\vec{r}_2 - \vec{r}_1)(\vec{dl}_1 \cdot \vec{dl}_2)}{|\vec{r}_2 - \vec{r}_1|^3}$$

The general motion of a charged particle in a homogeneous magnetic field is a helix: Uniform motion parallel to the magnetic field and circular motion perpendicular to it.

4: The magnetic dipole field

We know thet the far field of the magnetic and electric dipole look identical. We could calculate the magnetic dipole field from the Biot-Savart law, but that is tedious and involves several approximations for the far field. Instead we found that we could write the electric potential for the dipole as

$$V_{el} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$

If we define a magnetic dipole moment $\vec{m}$ can we use a similar potential to calculate the magnetic dipole field?

The magnetic dipole moment of a closed circuit with area $\vec{A}$ carrying a current I is

$$\vec{m} = I\vec{A}$$

Here we follow Duffin's notation and the "Sommerfeld convention". In other texts the "Kenelly convention" is used where $\vec{m} = \mu_0 I\vec{A}$.

The torque on a magnetic dipole $\vec{m}$ in a homogeneous magnetic field $\vec{B}$ is

$$\vec{T} = \vec{m} \times \vec{B}$$

The magnetic scalar potential would then become

$$V_{mag} = \frac{\mu_0}{4\pi} \frac{\vec{m} \cdot \vec{r}}{r^3}$$

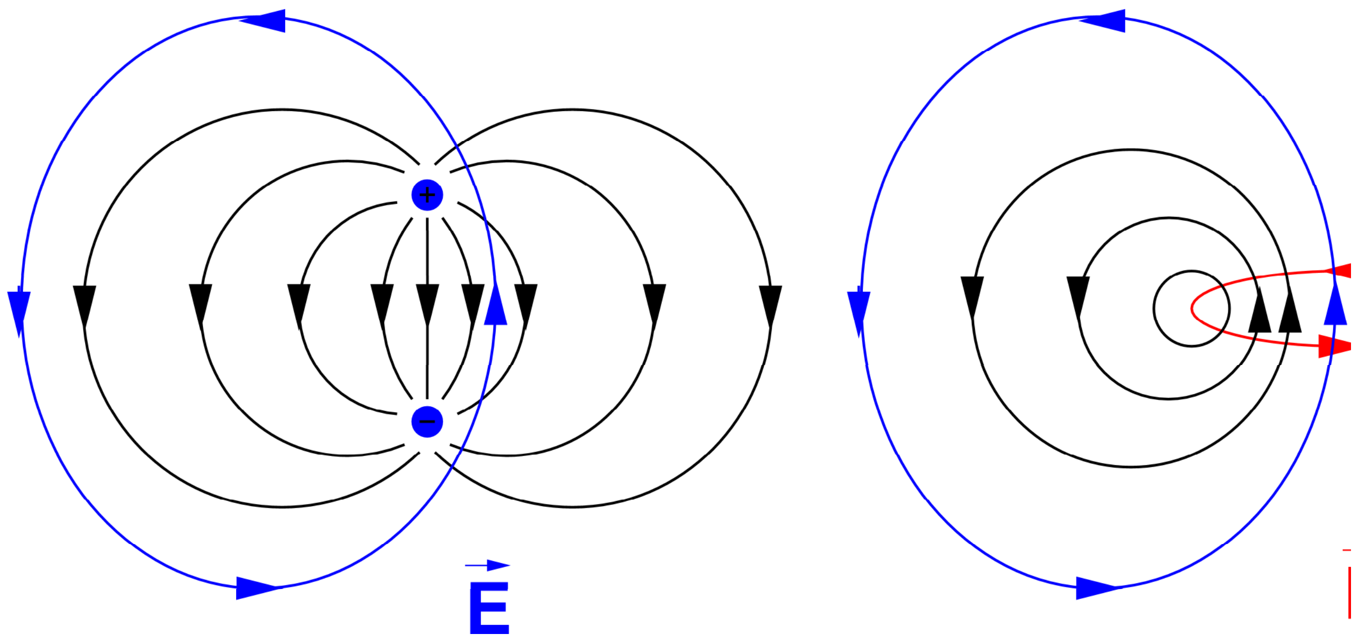
From this we instantly get the far-field of the magnetic dipole in cylinder coordinates:

$$\begin{aligned} B_z &= \frac{\mu_0 m}{4\pi} \frac{(3\cos^2\vartheta - 1)}{r^3} \\ B_\rho &= \frac{\mu_0 m}{4\pi} \frac{(3\cos\vartheta \sin\vartheta)}{r^3} \\ B_\phi &= 0 \end{aligned}$$

in complete analogy to the electric dipole.

Is V_{mag} a "good" potential? If so, the integral around any closed loop must vanish, and V must have a single value at each point.

For the electric dipole we know that is true. But look closely:



$$\oint_L \vec{E} \cdot \vec{dl} = 0 \qquad \oint_L \vec{B} \cdot \vec{dl} \neq 0!!$$

A general scalar potential does not exist for the magnetic dipole!

We have seen that any path that goes through the current loop has a non-zero contribution. What is it?

Take a special case: a long straight wire along the z-axis carrying a current I. We have earlier calculated the field to be

$$\vec{B}(x,y,z) = \frac{\mu_0 I}{2\pi(x^2+y^2)}(-y,x,0)$$

or, in cylinder coordinates

$$\vec{B}(\rho,\phi,z) = \frac{\mu_0 I}{2\pi\rho}\underline{\hat{\mathrm{a}}}_\phi$$

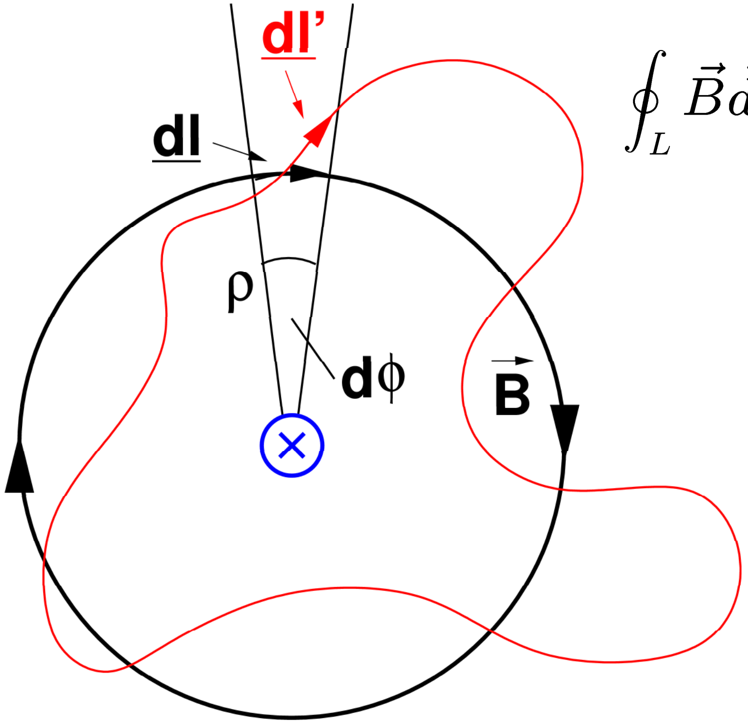
$\underline{\hat{\mathrm{a}}}_\rho, \underline{\hat{\mathrm{a}}}_\phi, \underline{\hat{\mathrm{a}}}_z$ form a right handed coordinate system

I) Take a circular path at constant distance ρ around the wire:

$$\oint_L \vec{B}\vec{dl} = \oint_L \frac{\mu_0 I}{2\pi\rho}\underline{\hat{\mathrm{a}}}_\phi \cdot \rho\, d\phi\, \underline{\hat{\mathrm{a}}}_\phi = \int_0^{2\pi} \frac{\mu_0 I}{2\pi}\, d\phi = \mu_0 I$$

II Take a general path around this wire:

$$\vec{dl'} = \rho(\phi)\, d\phi\, \hat{\underline{a}}_\phi + d\rho\, \hat{\underline{a}}_\rho + dz\, \hat{\underline{a}}_z$$



$$\oint_L \vec{B}\vec{dl'} = \int_0^{2\pi} \frac{\mu_0 I}{2\pi\rho(\phi)} \hat{\underline{a}}_\phi (\rho(\phi)\, d\phi\, \hat{\underline{a}}_\phi + d\rho\, \hat{\underline{a}}_\rho + dz\, \hat{\underline{a}}_z)$$

$$= \int_0^{2\pi} \frac{\mu_0 I}{2\pi\rho(\phi)} \rho(\phi)\, d\phi\, \hat{\underline{a}}_\phi \cdot \hat{\underline{a}}_\phi$$

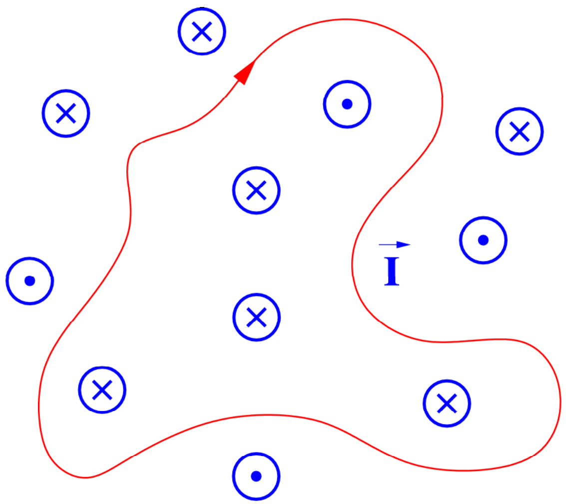
$$= \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = I\mu_0$$ independent of the path

This is Ampere's law in integral form

$$\oint_L \vec{B}\vec{dl} = \mu_0 I$$

By choosing $\oint d\vec{l} \rightarrow \int_0^{2\pi} d\phi$ we explicitly allow only one complete loop around the wire.

If we take many wires each carrying a current I their magnetic fields superimpose and the contribution to the line integral comes from all wires enclosed by the path



$$\rightarrow \oint \vec{B} d\vec{l} = 4\mu_0 I - \mu_0 I$$
$$= 3\mu_0 I$$

Duffin calls such closed paths enclosing a bit of current an Amperian path (in analogy to the Gaussian surface).

4.1: Ampere's law in differential form

We can easily transform this into a differential form:

$$\oint_L \vec{B}\vec{dl} = \int_S \text{curl}\,\vec{B}\,d\vec{S} = \mu_0 I = \mu_0 \int_S \vec{j}\,d\vec{S}$$

$$\rightarrow \text{curl}\,\vec{B} = \mu_0 \vec{j}$$

Compare this to the equivalent electric law:

$$\text{curl}\vec{E} = 0$$

The latter allowed us to define a scalar potential for electrostatic situations. The former makes it impossible to do the same for magnetostatics.

4.2: Examples

Wire of radius R containing constant current density $\vec{j} = j\,\underline{\hat{a}}_z$

For $\rho > R$ we know $\vec{B} = (\mu_0 I)/(2\pi\rho)\,\underline{\hat{a}}_\phi$

For $\rho \leq R$ we choose an Amperian path of radius ρ around the axis of the wire.

$$\oint \vec{B}\vec{dl} = \mu_0 \int_S \vec{j}d\vec{S} = \mu_0 j\pi\rho^2 \rightarrow |B| = \frac{\mu_0 j\pi\rho^2}{2\pi\rho} = \frac{\mu_0 j\rho}{2} = \frac{\mu_o I\rho}{2\pi R^2}$$

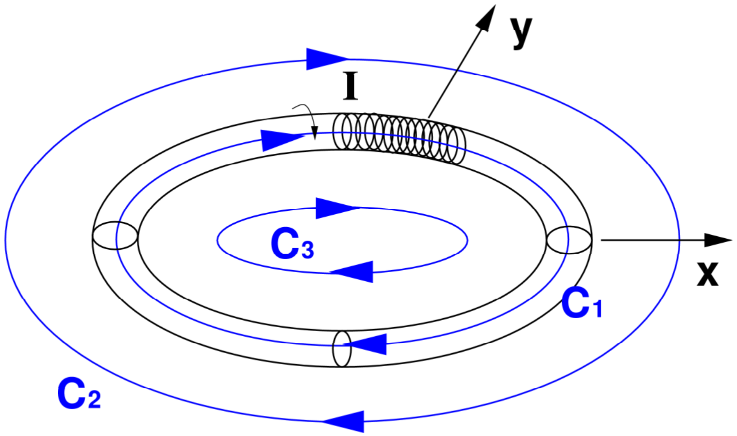
$$\vec{B} = \frac{\mu_o I\rho}{2\pi R^2}\,\underline{\hat{a}}_\phi$$

We now calculate the magnetic field of an toroidal solenoid with N coils, radius a carrying a current I
Path C_1 inside the solenoid:

$$\oint_{C_1} \vec{B}\,\vec{dl} = \mu_0 \int_S \vec{j}\,d\vec{S} \;=\; N\,I\,\mu_0$$

$$|\vec{B}| \cdot 2\pi R \;=\; N\,I\,\mu_0$$

$$\rightarrow |\vec{B}| \;=\; \frac{N\,I\,\mu_0}{2\pi R}$$



Explicitly

$$\vec{B} = \frac{N\,I\,\mu_0\,(+y, -x, 0)}{2\pi(x^2 + y^2)}$$

Outside (Paths C_2 and C_3):

$$\oint_{C_2} \vec{B}\vec{dl} = \oint_{C_3} \vec{B}\vec{dl} = 0 \rightarrow \vec{B} = 0$$

4.3: Magnetic flux through a *closed* surface

In Electrostatics we had

$$\oint_S \vec{E} d\vec{S} = \int_V \frac{\rho}{\epsilon_0} dV$$

What is the equivalent magnetic form?

If we take any number of electric dipoles fully enclosed by the surface S we know

$$\oint \vec{E} d\vec{S} = 0$$

because the net charge is zero. We also know that the far field for magnetic and electric dipoles is identical if we substitute

$$\vec{m} \leftrightarrow \vec{p} \quad \text{and} \quad \frac{1}{\epsilon_0} \leftrightarrow \mu_0$$

Thus for any surface enclosing complete magnetic dipoles

$$\oint \vec{B} \, d\vec{S} = 0$$

But we ran into trouble close to the dipoles earlier. Therefore we shall try a different approach.

We had earlier seen

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{j}\times\vec{r}}{|\vec{r}|^3}\,dV$$

We will write that as

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\vec{\nabla}\times\int_V \frac{\vec{j}}{|\vec{r}|}\,dV$$

Proof: We calculate

$$\begin{aligned}
\vec{\nabla}\times(\frac{\vec{a}}{|\vec{r}|}) &= \text{curl}\,\frac{\vec{a}}{|\vec{r}|} \\
&= (\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})\times(\frac{a_x}{\sqrt{x^2+y^2+z^2}},\frac{a_y}{\sqrt{x^2+y^2+z^2}},\frac{a_z}{\sqrt{x^2+y^2+z^2}})
\end{aligned}$$

$$\frac{\partial}{\partial x}\frac{a}{\sqrt{x^2+y^2+z^2}} = \frac{-x\,a}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{-x\,a}{r^3} \qquad \text{from tutorials}$$

$$\rightarrow \text{curl}\,\frac{\vec{a}}{r} = \left(-\frac{a_z\,y}{r^3} + \frac{a_y\,z}{r^3},\ -\frac{a_x\,z}{r^3} + \frac{a_z\,x}{r^3},\ -\frac{a_y\,x}{r^3} + \frac{a_x\,y}{r^3}\right)$$

$$= \frac{1}{r^3}\,(a_y\,z - a_z\,y,\ a_z\,x - a_x\,z,\ a_x\,y - a_y\,x)$$

$$= \frac{1}{r^3}\,(\vec{a} \times \vec{r})$$

$$\rightarrow \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{j} \times \vec{r}}{r^3} dV = \frac{\mu_0}{4\pi}\int \frac{\vec{\nabla} \times \vec{j}}{r} dV$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\vec{\nabla} \times \int_V \frac{\vec{j}}{r} dV$$

$\vec{B}$ can be written as the curl of something else. But since div curl $\vec{F} = 0$ for any $\vec{F}$

$$\rightarrow \quad \text{div}\vec{B} = 0$$

This is valid in the presence of all magnetic effects that can be described via the Biot Savart Law.

Starting from div$\vec{B} = 0$ we can derive an integral from via

$$0 = \int_V 0 dV = \int_V \text{div}\, \vec{B} dV = \oint_S \vec{B} d\vec{S}$$

$$\rightarrow \oint \vec{B} d\vec{S} = 0 \qquad \text{for any surface}$$

$\rightarrow$ Magnetic field lines <u>must</u> form closed loops, they do not begin or end anywhere. There are no sources or drains of magnetic field lines!

This is a complete set of Maxwell's equations for steady conditions in vacuum:

in integral form:

$$\rightarrow \oint_S \vec{B}\, d\vec{S} = 0$$

$$\oint_L \vec{B}\, d\vec{l} = \mu_0 I$$

$$\oint_S \vec{E}\, d\vec{S} = \frac{Q}{\epsilon_0}$$

$$\oint_L \vec{E}\, d\vec{l} = 0$$

in differential form:

$$\text{div}\, \vec{B} = 0$$

$$\text{curl}\, \vec{B} = \mu_0\, \vec{j}$$

$$\text{div}\, \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\text{curl}\, \vec{E} = 0$$

In addition we had these important equations:

$$\text{div}\vec{j} = \frac{-d\rho}{dt}$$

$$\vec{F}_l = q(\vec{E} + \vec{v} \times \vec{B})$$

$$\vec{j} = \sigma\, \vec{E}$$

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{l} \times \vec{r}}{r^3}$$

$$\vec{F}_c = \frac{q \cdot q_2}{4\pi\epsilon_0} \frac{\vec{r}}{|r|^3}$$

4.4: The Vector Potential

It was very useful to have a potential for $\vec{E}$. We already know that it is generally impossible to define a similar scalar potential for $\vec{B}$.

Electric

$$\vec{E} = \int_V \frac{\rho(r)\vec{r}}{4\pi\epsilon_0|r|^3}\,dV = -\vec{\nabla}\cdot\int_V \frac{1}{4\pi\epsilon_0}\frac{\rho}{r}\,dV$$

$$V_{el} = \int_V \frac{\rho}{4\pi\epsilon_0 r}\,dV$$

$$\vec{E} = -\vec{\nabla}\,V$$

Magnetic

$$\vec{B} = \int_V \frac{\mu_0\vec{j}\times\vec{r}}{4\pi r^3}\,dV = \vec{\nabla}\times\int_V \frac{\mu_0}{4\pi}\frac{\vec{j}}{r}\,dV$$

$$\vec{A}_{mag} = \int_V \frac{\mu_0}{4\pi}\frac{\vec{j}}{r}\,dV$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$\vec{A}$ is the magnetic vector potential

V was not unique: $V' = V + const$ gave the same $\vec{E}$.

$\vec{A}$ is not unique either:

$$\begin{aligned}\vec{A}' &= \vec{A} + \vec{F} \\ \vec{\nabla} \times \vec{A}' &= \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{F}\end{aligned}$$

The last part vanishes if an $\vec{F} = \vec{\nabla}\,\Phi$ for any scalar field Φ. Thus we can add the gradient of any scalar field to $\vec{A}$ and obtain an alternative vector potential that gives the same magnetic field $\vec{B}$:

$$\begin{aligned}\vec{\nabla} \times \vec{A}' &= \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{F} \\ &= \vec{B} + \vec{\nabla} \times (\vec{\nabla}\,\Phi) \\ &= \vec{B} + 0\end{aligned}$$

The vector potential now will offer similar advantages to calculations with magnetic fields as the electric potential.

$$\vec{B} = B_0\,\underline{\hat{\mathsf{a}}}_z = \mathrm{curl}\vec{A}$$

$$\begin{aligned}
\frac{\partial}{\partial\,y}A_z - \frac{\partial}{\partial\,z}A_y &= 0\\
\frac{\partial}{\partial\,z}A_x - \frac{\partial}{\partial\,x}A_z &= 0\\
\frac{\partial}{\partial\,x}A_y - \frac{\partial}{\partial\,y}A_x &= B_0
\end{aligned}$$

$$A_y = B_0\,x \qquad A_x = A_z = 0 \qquad \vec{A}_1 = (0, x\,B_0, 0)$$

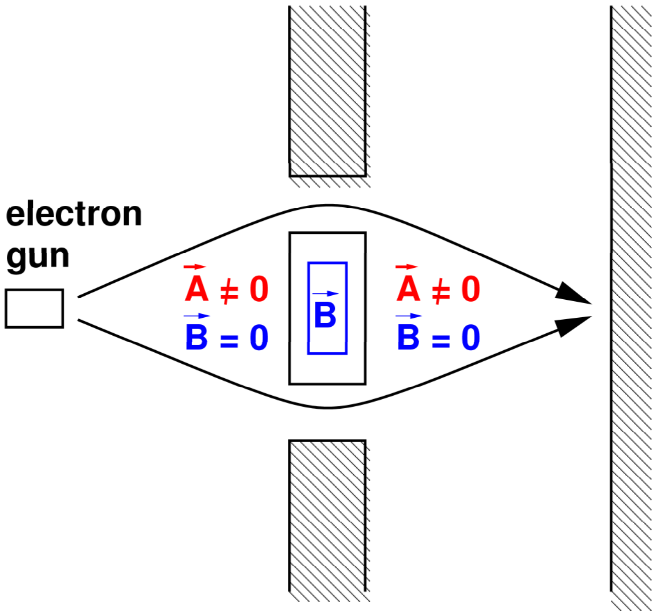
or $$A_x = -B_0\,y \quad A_y = A_z = 0 \qquad \vec{A}_2 = (-y\,B_0, 0, 0)$$

or $$A_x = -\frac{1}{2}y\,B_0 \quad A_y = \frac{1}{2}x\,B_0 \quad A_z = 0 \qquad \vec{A}_3 = (\frac{-y\,B_0}{2}, \frac{x\,B_0}{2}, 0)$$

$$\vec{A}_3 = \frac{1}{2}\vec{B} \times \vec{r}$$

$$\vec{A}_1 = \vec{A}_2 + \vec{\nabla}\,\Phi \qquad \text{if}\,\Phi = B_0\,x\,y$$

Is it real? Yes! But the experiment is difficult. 1957 the Aharonov-Bohm effect was experimetally demonstrated.



Take a double slit and fire electrons at it. The matter waves will diffract and form a double-slit pattern behind the slits.

Now we introduce a magnetic field in the space between the slits such that it is zero everywhere where electrons can travel, i.e. a toroidal solenoid

The electrons cannot interact with $\vec{B}$, but if they interact with $\vec{A}$, we should see a change in the interference pattern.

$\rightarrow$ Just as V_{el} is real, $\vec{A}$ is real!

Its dimension is Tm = V s / m = N / A

4.5: Summary

The complete set of Maxwell's equations for steady conditions in vacuum:

in integral form:

$$\rightarrow \oint_S \vec{B}\, d\vec{S} = 0$$

$$\oint_L \vec{B}\, d\vec{l} = \mu_0 I$$

$$\oint_S \vec{E}\, d\vec{S} = \frac{Q}{\epsilon_0}$$

$$\oint_L \vec{E}\, d\vec{l} = 0$$

in differential form:

$$\text{div}\, \vec{B} = 0$$

$$\text{curl}\, \vec{B} = \mu_0\, \vec{j}$$

$$\text{div}\, \vec{E} = \frac{\rho}{\epsilon_0}$$

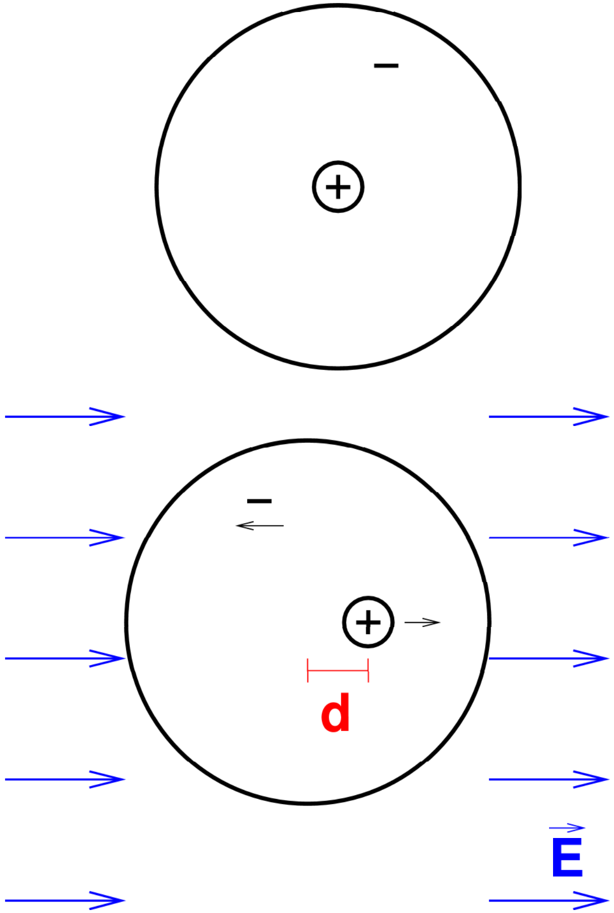
$$\text{curl}\, \vec{E} = 0$$

In addition to the electric potential we defined a magnetic vector potential:

$$\vec{A}_{mag} = \int_V \frac{\mu_0}{4\pi} \frac{\vec{j}}{r}\, dV$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

4.6: Dielectrics

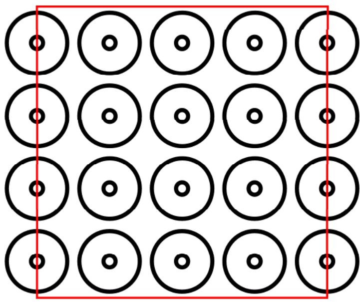


We will now investigate what happens if we introduce a piece of insulating material into an electric field $\vec{E}$. The material is made up of charged particles, positive nuclei and negative electrons, thus we expect some influence on the material.

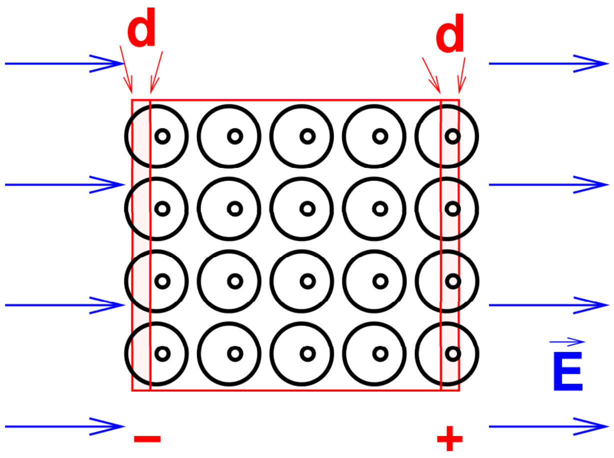
Consider a neutral atom with Z electrons and Z protons.

We obtain an electric dipole moment $\vec{p} = Z\,e\,d\,\underline{\hat{\mathrm{a}}}_E$ where $\underline{\hat{\mathrm{a}}}_E$ is a unit vector in the direction of $\vec{E}$.

If the insulator has a density of N atoms per unit volume the total electric dipole moment per unit volume becomes $\vec{p} = N\,Z\,e\,d\,\underline{\hat{\mathrm{a}}}_E$.

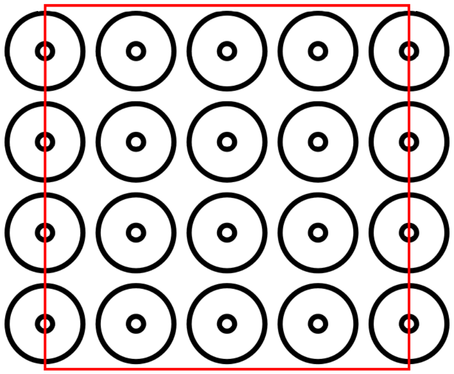


As *all* negative charges move against $\vec{E}$ and *all* positive charges move with $\vec{E}$, we build up surface charges: Consider a small unit volume inside the insulator.



It's net charge is zero. If an external electric field is applied, its net charge remains zero. But consider the small strips of width d at either end of the volume: If one near the entrance of $\vec{E}$ is to remain neutral then atomic nuclei have to have to move into it. If this strip is at the surface of the material that is impossible and it will contain a net negative charge.

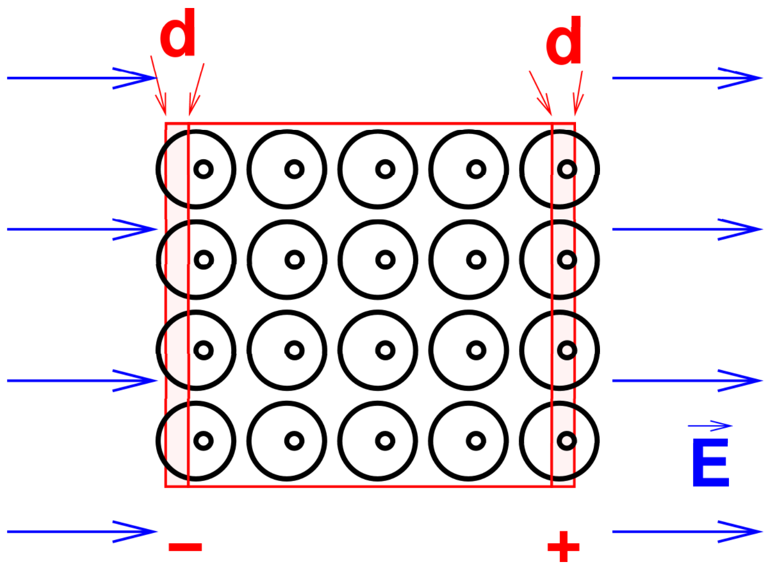
Similarly on the other side. Electrons have been sucked out of this strip, and at the surface of the material they cannot be replenished, therefore we build up a net positive charge.



How big are these charges?

The shaded small volumes have a thickness d and a surface area dS. The charge density of electrons was $-N\,Z\,e$, thus the total charge in this volume is

$$-N\,Z\,e\,d\,dS = -N\,p\,dS$$



This is equal to a surface charge density

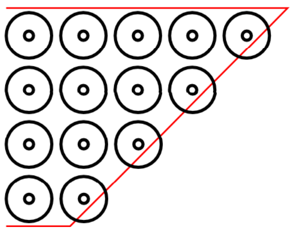
$$\sigma = -N\,p$$

Similarly the other surface acquires a surface charge density

$$\sigma = +N\,p$$

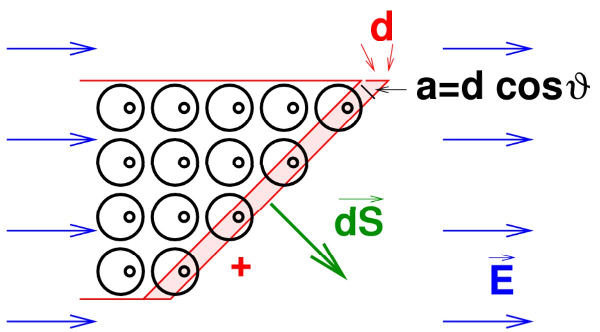
If the surface $\underline{dS}$ is not perpendicular to the direction of the $\vec{E}$ we still have

$$\vec{p} = N\,Z\,e\,d\,\underline{\hat{\mathsf{a}}}_E$$



The total charge in a surface element of thickness $a = d\cos\vartheta$ is

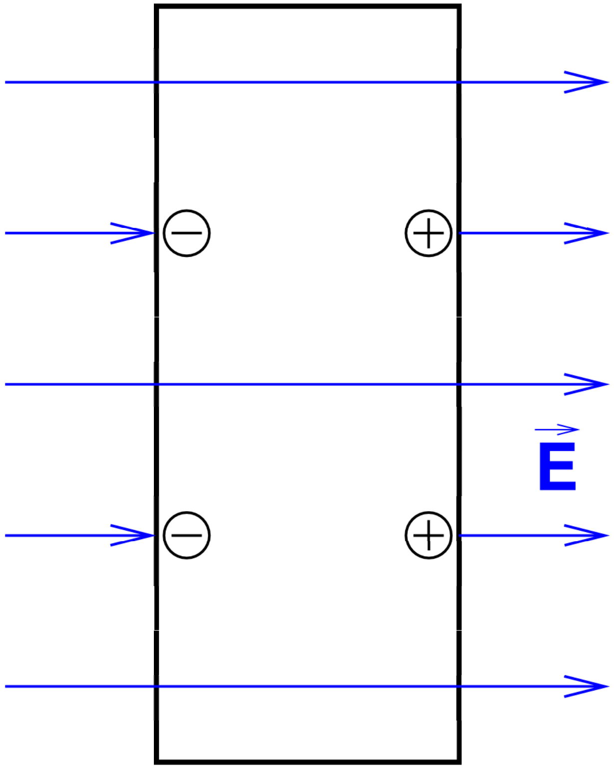
$$\begin{aligned} N Z\, e\, a\, dS &= N Z e\, d \cos\vartheta\, dS \\ &= N\vec{p}\,\underline{dS} = \vec{P}\,\underline{dS} \end{aligned}$$



$\vec{P}$ is the dipole moment per unit volume of the material. We call $\vec{P}$ the polarization. The surface charge density can then be written $\sigma_p = \pm P$. The polarization will depend strongly on the material:

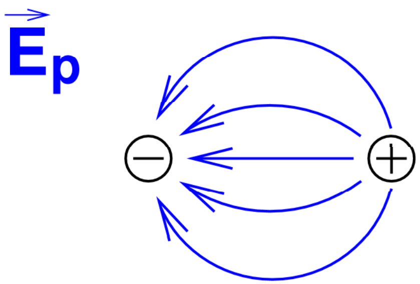
$$\vec{P} = \chi_e \epsilon_0 \vec{E}$$

χ_e is called the electric susceptibility of the material and is dimensionless.

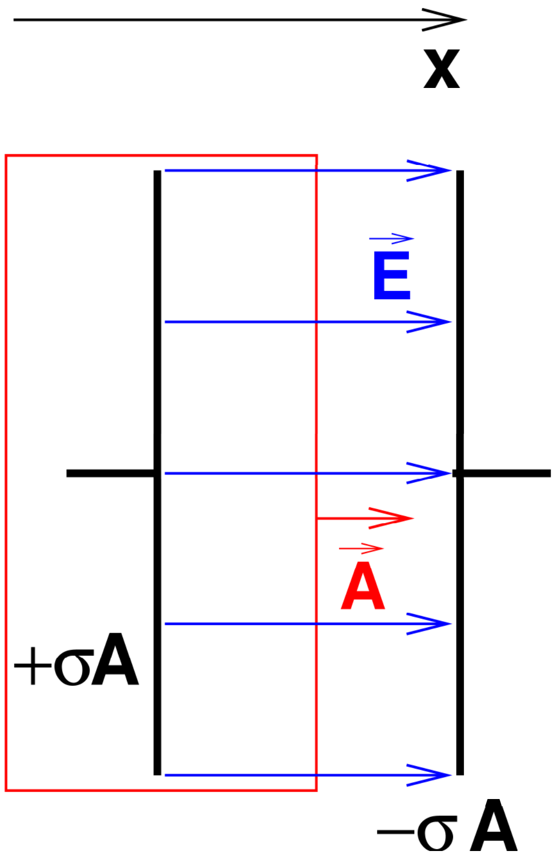


In general the same arguments apply to the susceptibility as to the conductivity: It need not be a scalar constant. Generally it will be a tensor. In this course we will again only consider media in which χ_e is a real number: Linear, isotropic and homogeneous (LIH) materials.

We shall now examine a plate capacitor with a dielectric material more closely. First we look at the inside of a slab of dielectric in a electric field:



The electric field inside the dielectric is reduced because field lines can now start and end on the induced surface polarization charges. If you look more closely you will note that the electric field from *each* dipole is oriented *opposite* the external field. The net field is the sum of both, and we get a reduction of $\vec{E}$.



We can now introduce this slab into a plate capacitor with two parallel plates a distance d apart with surface area A.

Without this dielectric slab we have a charge $Q = \pm\sigma A$ on the plates and the electric field inside is constant: $E_0 = \sigma/\epsilon_0$ from Gauss' Law:

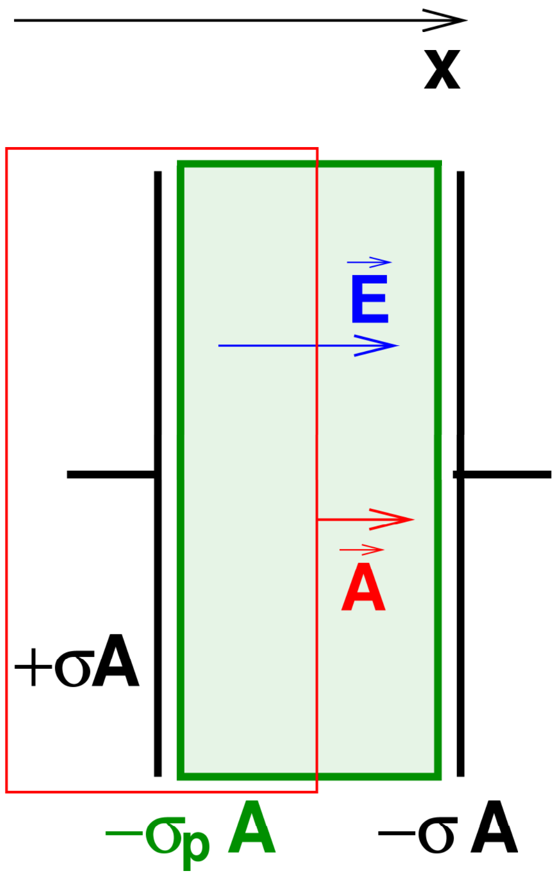
$$\oint_S \vec{E}\cdot d\vec{S} = \vec{E}\cdot\vec{A} = E_0\underline{\hat{\mathsf{a}}}_x A\underline{\hat{\mathsf{a}}}_x = \frac{\sigma A}{\epsilon_0} \rightarrow = E_0 = \frac{\sigma}{\epsilon_0}$$

$$\Delta V = -\int_0^d \vec{E}\underline{\mathsf{dl}} = -\int_0^d E_0\,\underline{\hat{\mathsf{a}}}_x\cdot(-dx\,\underline{\hat{\mathsf{a}}}_x) = E_0 d$$

And the capacitance

$$C_0 = \frac{Q}{\Delta V} = \frac{\sigma A}{E_0 d} = \frac{\sigma A}{\frac{\sigma}{\epsilon_0}d}$$

$$C_0 = \frac{\epsilon_0 A}{d}$$



Now the slab of dielectric enters the space between the plates leaving a tiny gap on either side. We keep the potential difference fixed at ΔV. Polarization charges appear and in the vast majority of the volume the electric field is reduced:

The Gaussian surface now encloses the charge $(\sigma - \sigma_p)A$ which leads to an electric field $E = (\sigma - \sigma_p)/\epsilon_0 = (\sigma - P)/\epsilon_0$

If we keep ΔV constant (via a battery) the σ must change:

$$\sigma = \epsilon_0 E + P = \epsilon_0 E + \chi_e \epsilon_0 E = \epsilon_0 (1 + \chi_e) E$$

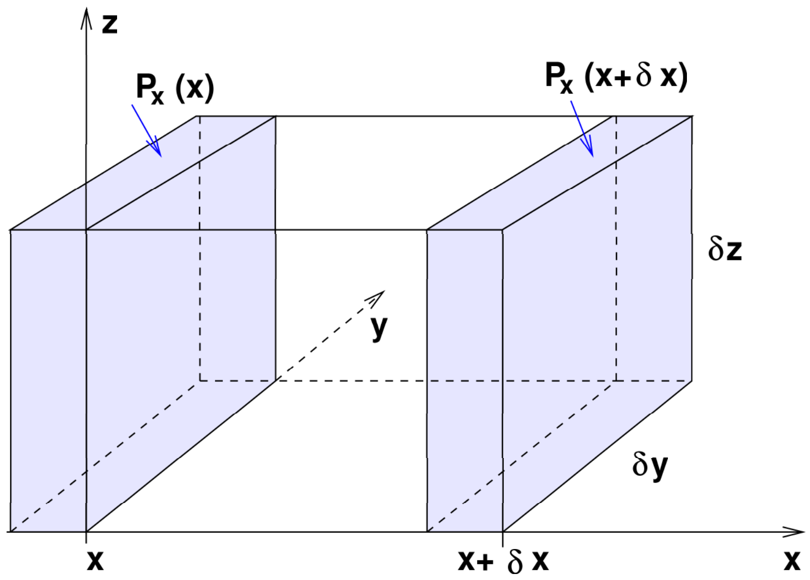
The capacitance is now

$$\begin{aligned} C &= \frac{\sigma A}{\Delta V} \\ &= \frac{\epsilon_0(1+\chi_e)AE}{Ed} \\ &= C_0(1+\chi_e) \end{aligned}$$

We call $(1+\chi_e) = \epsilon_r$ the relative permittivity and write $C = (A/d)\epsilon_0\epsilon_r$.

Introducing a dieletric into a capacitor increases the capacity by $\epsilon_r = (1+\chi_e)$.

Thus we could also write $\vec{P} = \epsilon_0(\epsilon_r - 1)\vec{E}$

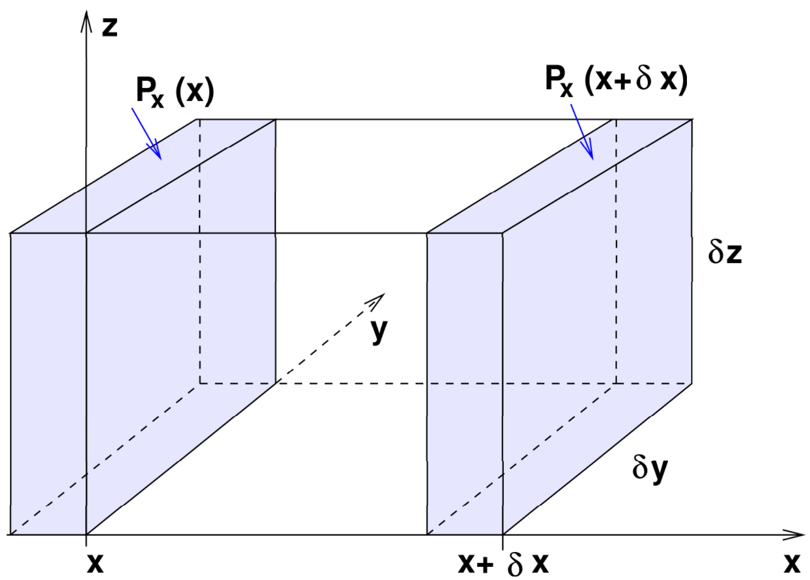


We can now turn to the more general case of dieletric material in a non-uniform eletric field. In addition to surface charges σ_p we might also get a volume charge distribution ρ_p.

Imagine this small cube inside a neutral dielectric. Suppose it is placed in an electric field so that it acquires a polarization $\vec{P}$. If the electric field is non-uniform it will result in a non-uniform polarization.

Assume that P_x changes by δP_x over the length of the cube δx, we get for the charge flowing into the left hand side $P_x(x)\delta y\delta z$ and for the charge flowing out the other side $P_x(x+\delta x)\delta y\delta z$. Thus the net charge flowing into the cube is

$$[P_x(x)\,\delta y\delta z - P_x(x+\delta x)\,\delta y\delta z] = -\left[\frac{P_x(x+\delta x)-P_x(x)}{\delta x}\,\delta x\right]\delta y\delta z = -\frac{\partial P_x}{\partial x}\,\delta x\delta y\delta z$$



Similar arguments apply to the other faces and we get for the net polarization charge acquired by cube via all faces:

$$-\left(\frac{\partial P_x}{\partial x}+\frac{\partial P_y}{\partial y}+\frac{\partial P_z}{\partial z}\right)\delta x\delta y\delta z=\rho_p\,\delta x\delta y\delta z$$

In the entire material we find a macroscopic polarization charge

$$\rho_p=-\vec{\nabla}\cdot\vec{P}$$

Is this material still neutral? We can add all surface and volume charges to obtain

$$\oint_S\sigma_p\,dS+\int_V\rho_p\,dV=\oint_S\vec{P}d\vec{S}-\int_V \mathrm{div}\vec{P}\,dV=0$$

The last step uses Gauss' theorem.

4.7: The electric displacement field $\vec{D}$

In the most general case, the dielectric may contain both polarization charges ρ_p and free charges ρ_f, i.e. carry a net charge.

All these charges need to be considered in Gauss' law:

$$\begin{aligned} \mathrm{div}\vec{E} &= \frac{\rho}{\epsilon_0} = \frac{\rho_f + \rho_p}{\epsilon_0} \\ &= \frac{\rho_f - \mathrm{div}\vec{P}}{\epsilon_0} \end{aligned}$$

We can rearrange this as

$$\epsilon_0\,\mathrm{div}\vec{E} + \mathrm{div}\vec{P} = \rho_f$$

We define the electric displacement $\vec{D}$:

$$\vec{D} = \epsilon_0\,\vec{E} + \vec{P}$$

And Gauss' law in the presence of dielectrics takes the simple shape

$$\text{div}\vec{D} = \rho_f$$

In $\vec{D}$ all polarization effects are already taken into account.

The units for $\vec{D}$ and $\vec{P}$ are C/m^2.

We can also write

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon_0\vec{E} + \chi_e\epsilon_0\vec{E} = \epsilon_0(1 + \chi_e)\vec{E} = \epsilon_0\epsilon_r\vec{E}$$

In integral form Gauss' law in the presence of dielectrics becomes

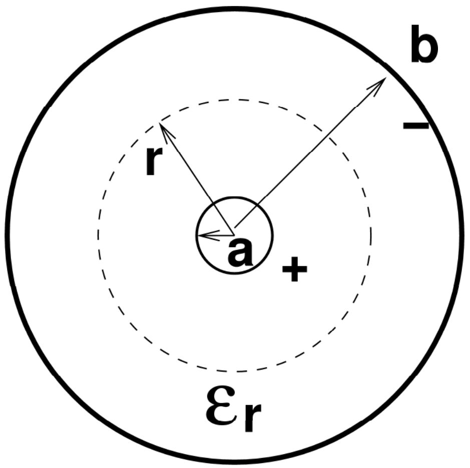
$$\int_V \text{div}\vec{D}\, dV = \oint_S \vec{D}d\vec{S} = \int_V \rho_f\, dV$$

The flux of $\vec{D}$ through any closed surface is equal to the net enclosed charge.

4.8: Example

We calculate the capacitance per unit length of a coaxial cable consisting of concentric cylindrical conducturs of radii a and b filled with a dielectric of relative permittivity ϵ_r.

A cylindrical Gaussian surface allows us to calculate the magnitude of $\vec{D}$ between the cylinders:



$$\oint_S \vec{D} d\vec{S} = 2\pi r h D(r) = 2\pi a h \sigma_f$$

$$D(r) = \frac{a\sigma_f}{r}$$

We can use symmetry arguments to find that $\vec{D}$ is radial:

$$\vec{D}(r) = \frac{a\sigma_f}{r} \underline{\hat{\mathsf{a}}}_r$$

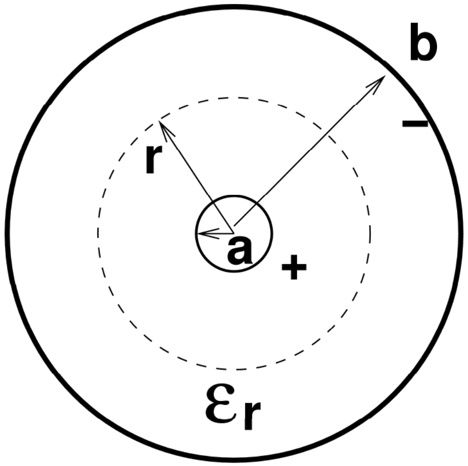
Or

$$\vec{E}(r) = \frac{a\sigma_f}{\epsilon_0 \epsilon_r r} \underline{\hat{\mathsf{a}}}_r$$

The potential difference becomes

$$\Delta V = \int_a^b \frac{a\,\sigma_f}{\epsilon_0\epsilon_r\, r}\, dr = \frac{a\,\sigma_f}{\epsilon_0\epsilon_r}\ \ln(b/a)$$

Finally we get the capacitance



$$C = \frac{Q}{\Delta V} = \frac{\sigma_f 2\pi a L}{\frac{a\sigma_f}{\epsilon_0\epsilon_r}\ \ln(b/a)}$$

$$= \frac{2\pi\epsilon_0\epsilon_r L}{\ln(b/a)}$$

Finally the capacitance per unit length is

$$C' = \frac{C}{L} = \frac{2\pi\epsilon_0\epsilon_r}{\ln(b/a)}$$

4.9: Behaviour at Boundaries

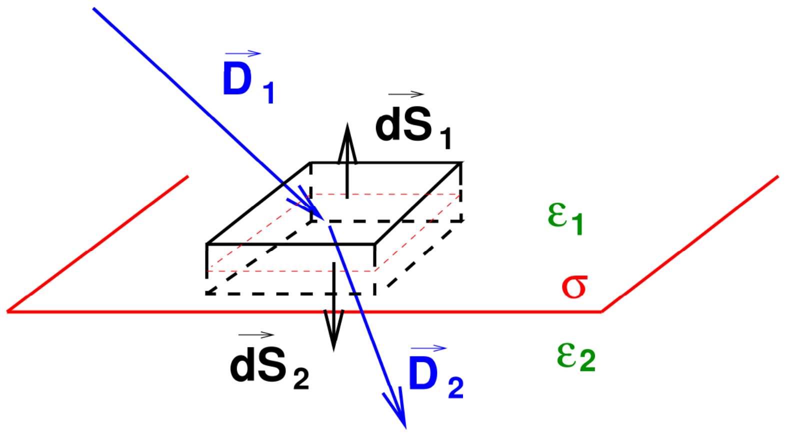
We now study the behaviour of $\vec{E}$ and $\vec{D}$ at the boundary between two dielectrics with relative permittivities ϵ_1 and ϵ_2. The boundary can also carry a net surface charge σ.

Consider the volume enclosed by the small cylinder. We know

$$\oint \vec{D}\, d\vec{S} = \int_V \sigma\, dV = \vec{D}_1 \cdot d\vec{S}_1 + \vec{D}_2 \cdot d\vec{S}_2 = \sigma dS$$

$$D_{2\perp}\, dS - D_{1\perp}\, dS = \sigma\, dS$$

$$\Leftrightarrow \; D_{2\perp} - D_{1\perp} = \sigma$$

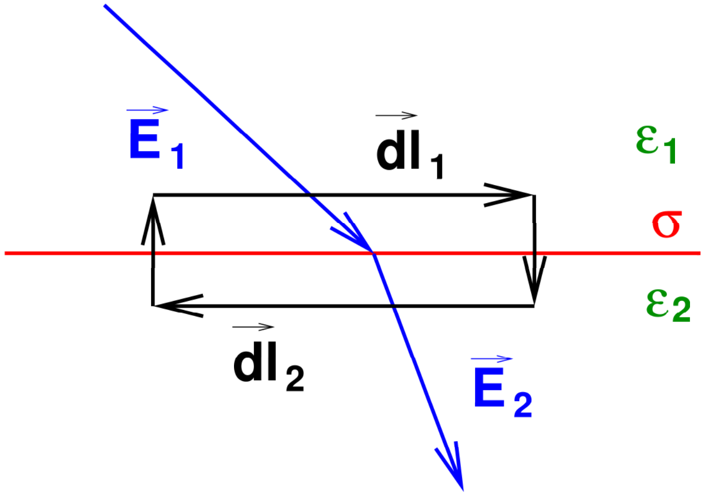


The normal component of $\vec{D}$ across a boundary is discontinuous by σ.

If $\sigma = 0$ the normal component of $\vec{D}$ across a boundary is continuous.

The components parallel to the surface can be investigated via the circuital law:

$$\oint_L \vec{E}\,\vec{dl} = 0$$



$$\vec{E}_1\,\vec{dl}_1 + \vec{E}_2\,\vec{dl}_2 = (E_{1\|} - E_{2\|})\,dl = 0$$

The components of $\vec{E}$ parallel to the boundary between two dielectrics are continuous.

4.10: Summary

The polarization is given by

$$\vec{P} = \chi_e \epsilon_0 \vec{E}$$

χ_e is called the electric susceptibility of the material and is dimensionless.

The capacitance of a plate capacitor with dielectric is

$$C = \frac{A\,\epsilon_0 \epsilon_r}{d}$$

We call $(1 + \chi_e) = \epsilon_r$ the relative permittivity

The polarization is related to a polarization volume charge density via

$$\text{div}\,\vec{P} = -\rho_p$$

We define the electric displacement $\vec{D}$:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

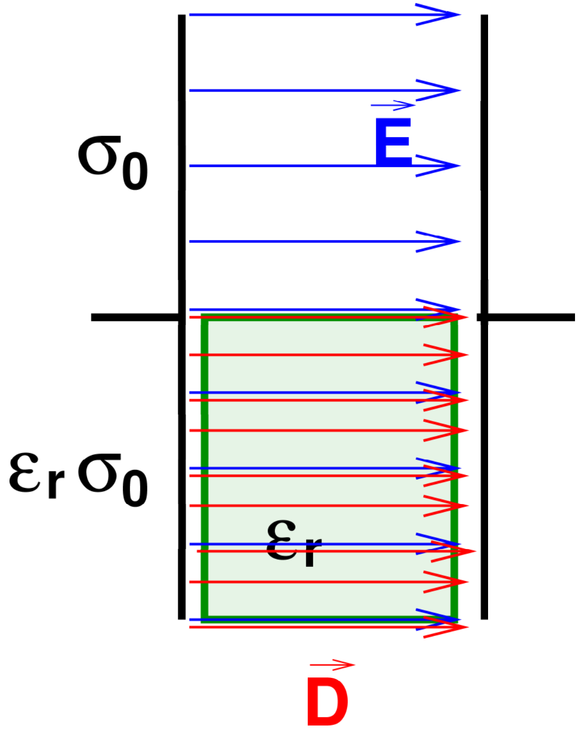
And Gauss' law in the presence of dielectrics takes the simple shape

$$\text{div}\vec{D} = \rho_f \qquad \oint_S \vec{D} d\vec{S} = \int_V \rho_f \, dV$$

The normal component of $\vec{D}$ across a boundary is discontinuous by σ.

The components of $\vec{E}$ parallel to the boundary between two dielectrics are continuous.

4.11: Force on a dielectric



If we take a plate capacitor and partially fill it with a dielectric, what will happen?

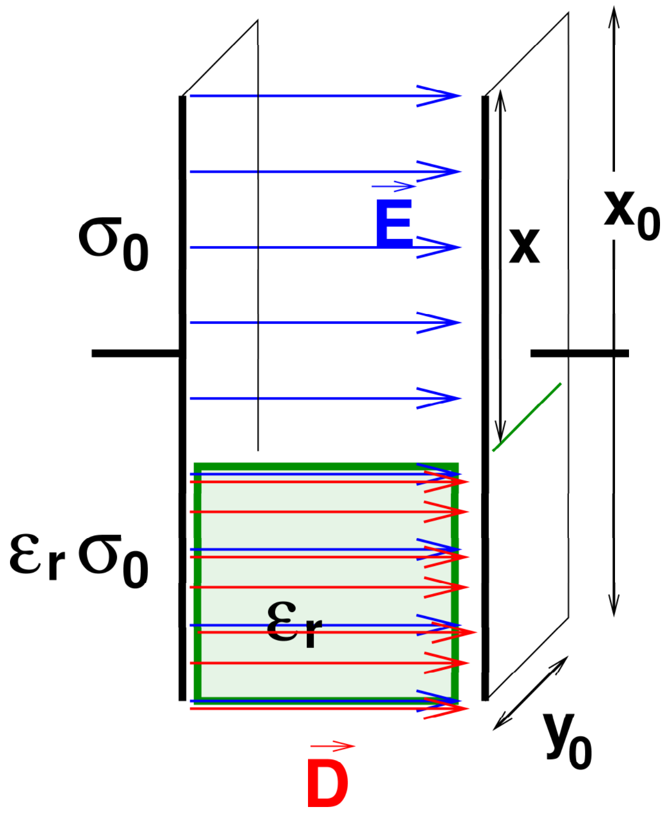
- Experiment shows that the dielectric is pulled into a region of high electric field.
- To see why, we will calculate the field and energy in a capacitor partially filled with a dielectric.
- Then we find the force on the dielectric as the negative gradient of the energy.

The energy of a capacitor comes from the buildup of charge and is supplied by an external source (battery). The energy needed to store additional charge dQ onto a capacitor plate at potential V' is $V'\,dQ$.

$$U_{pot} = \int_0^Q V' dQ'$$

Using $C = Q/V \quad , \quad dQ = CdV \quad$ and $C = \epsilon_0 \epsilon_r A/d \quad$ for a plate capacitor we rewrite

$$U_{pot} = \int_0^V V' C dV' = \frac{1}{2} C V^2 = \frac{1}{2} \frac{Q^2}{C}$$



Now we must deduce the exact field configuration in the capacitor if it is partially filled with a dielectric.

The rectangular plate has an area $A = x_0 \cdot y_0$. The length of vacuum (or air) in the capacitor is x.
We can treat this as two parallel plate capacitors, C_1 with dielectric and C_2 without it. We get

$$C_1 = \frac{\epsilon_0 \epsilon_r (x_0 - x) y_0}{d} \qquad C_2 = \frac{\epsilon_0\, x\, y_0}{d}$$

The total capacitance is just the sum of the two:

$$C = C_1 + C_2 = \frac{\epsilon_0\, y_0}{d}[\epsilon_r(x_0 - x) + x] = \frac{\epsilon_0\, y_0}{d}[\epsilon_r x_0 - \chi_e x]$$

We shall first assume that the capacitor is disconnected from a battery and therefore holds a constant charge Q_0. The energy stored in the capacitor is

$$U = \frac{1}{2}\frac{Q_0^2}{C}$$

We need to calculate the force on the dielectric, which we get as the negative gradient of the energy:

$$\vec{F} = -\nabla U \qquad \text{or, just in x-direction} \qquad F = -\frac{dU}{dx}$$

$$F = -\frac{dU}{dx} = -\frac{dU}{dC}\frac{dC}{dx} = \frac{1}{2}\frac{Q_0^2}{C^2}\frac{dC}{dx} = \frac{V^2}{2}\frac{dC}{dx}$$

$$\frac{dC}{dx} = \frac{d}{dx}\frac{\epsilon_0\, y_0}{d}[\epsilon_r x_0 - \chi_e x] = -\frac{\epsilon_0\, y_0\, \chi_e}{d}$$

So we get the force on the dielectric as

$$F = -\frac{V^2 \epsilon_0\, y_0\, \chi_e}{2d}$$

into the capacitor!

What happens if we keep the potential constant, instead of the charge on the plates?

If we connect an external battery it will also do work as the charges rearrange themselves in the presence of the dielectric. This extra work must be taken into account.

Without the work done by the battery we get:

$$F = -\frac{d}{dx}U = -\frac{d}{dx}\frac{1}{2}V^2 C = -\frac{V^2}{2}\frac{dC}{dx}$$

which seems to imply that the dielectric is pushed out!

But if we add the work done by the battery on the charges $W_{batt} = \int V \, dQ$ we get

$$F = -\frac{d}{dx}W_{tot} = -\frac{V^2}{2}\frac{dC}{dx} + V\frac{dQ}{dx} = -\frac{V^2}{2}\frac{dC}{dx} + V^2\frac{dC}{dx} = +\frac{V^2}{2}\frac{dC}{dx}$$

as before. The dielectric is still pulled into the capacitor.

4.12: Magnetic Materials

The spectrum of magnetic materials is far larger and more varied than that of dielectrics. We distinguish

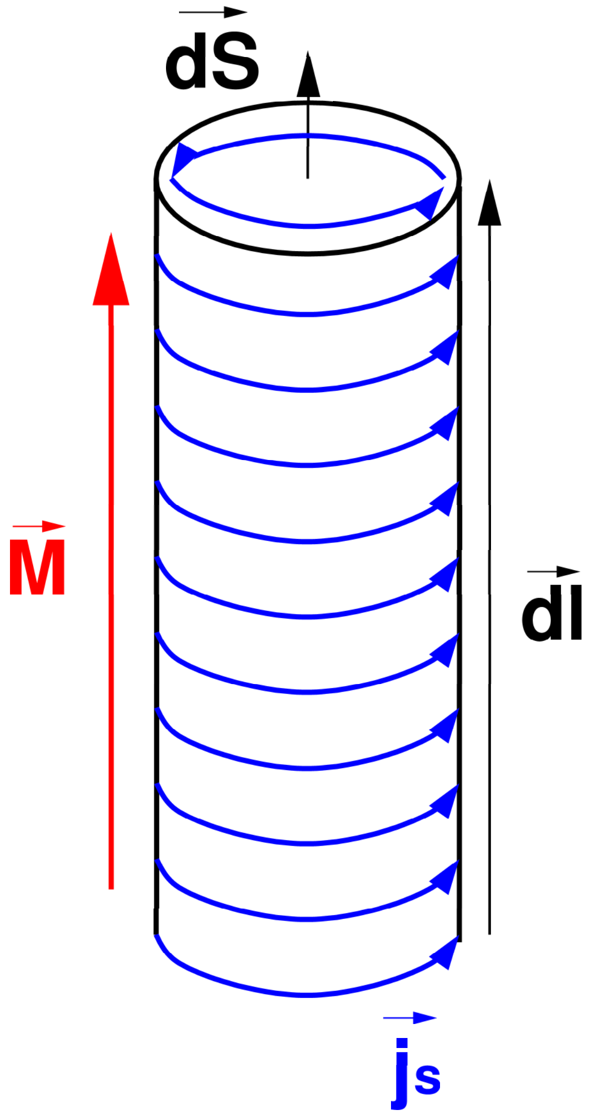
diamagnetic	Carbon, Silicon, Plastic, Glass
paramagnetic	Sodium, Bromine, Aluminium
ferromagnetic	Iron, Nickel, Cobalt, Gadolinium
antiferromagnetic	WO_3
ferrimagnetic	Certain iron oxides

These in principle need individual treatment and whole branches of solid state physics are devoted to their study. In the previous chapter we were able to reduce all material specific properties to a single numeric constant $\chi_e = \epsilon_r - 1$. Can we repeat this for magnetic materials?

Antiferro- and ferrimagnetics will not be considered. Ferromagnets show hysteresis which allows the formation of permanent magnets, but as we shall see also means that the magnetisation of the material is not just a function of the external field, but also of the history - We will describe such behaviour, but not treat it in an exact manner. That leaves diamagnetic and paramagnetic materials.

If an external magnetic field is applied to a slab of matter, two things happen: small current loops are induced in the material analogous to the creation of dipoles in a dielectric. In addition the atoms in the material already may form little magnetic dipoles which get aligned in a field. In both cases the net effect is the creation of a macroscopic magnetisation $\vec{M}$ of the material.

We will now follow Duffin's treatment. First we deduce how the magnetisation is related to Amperian currents in the surface of the material. Then we define the magnetic field strength $\vec{H}$ and relate it to $\vec{M}$ and $\vec{B}$ via Ampere's Law. Finally we introduce appropriate material constants μ_r and χ_m.



Take a small cylindrical piece of material with length dl and cross section dS. Let it possess a magnetisation $\vec{M}$. The magnetic dipole moment is then

$$\vec{M}\,dV = \vec{M}dl\,dS$$

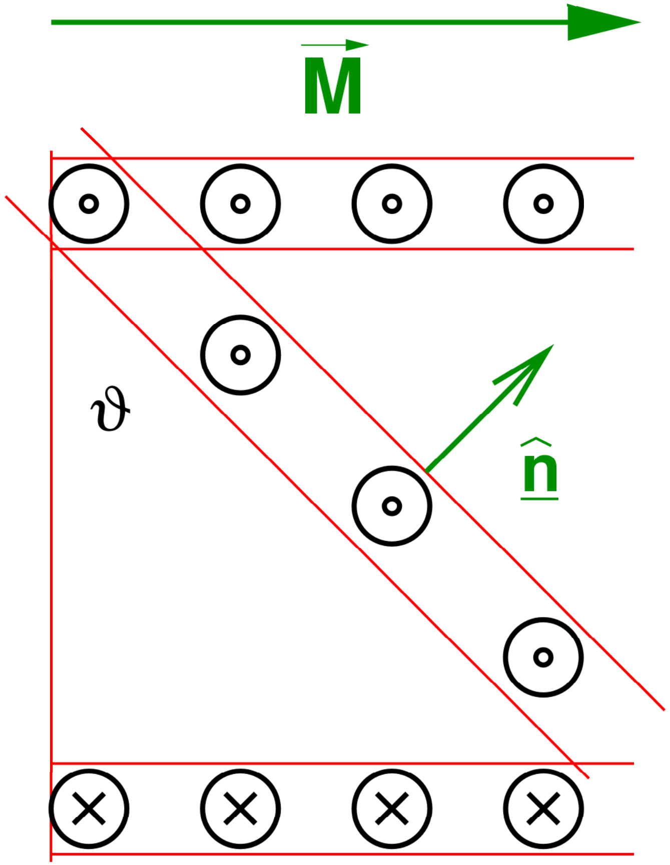
The magnetic dipole moment of the indicated surface current density $\vec{j}$ is

$$|\vec{j}_S|\underline{dl}\,dS$$

If $\vec{M}$ is parallel to $\underline{dl}$ then we find

$$|\vec{j}_S| = |\vec{M}|$$

If we make the block out of many such elements, then all current densities at boundaries inside the block cancel, and we are left with only surface currents.



What if the surface is not parallel to $\vec{M}$?

j_S remains the same, but the surface area increases. Thus the surface density decreases by $\sin\delta$ to

$$j_S = M\sin\delta = |\widehat{M}\times\widehat{n}|$$

In addition, if $\vec{M}$ is not uniform we can have Amperian currents inside the material given by

$$\vec{j}_m = \vec{\nabla}\times\vec{M}$$

We shall denote all these Amperian currents with I_M If we introduce a slab of magnetic mat erial into a region of space already containing a magnetic field $\vec{B}_0$ it will become nagetised and we get Amperian currents I_M. These in turn give rise to an external magnetic field $\vec{B}_M$ via Biot-Savart's Law.

We obtain $\vec{B} = \vec{B}_0 + \vec{B}_M$ as the magnetic field everywhere and investigate if and how our field equations must be modified.

$$\text{div}\vec{B} = \text{div}\vec{B}_0 + \text{div}\vec{B}_M = 0 + 0 = 0 \quad \leftrightarrow \quad \oint_S \vec{B} \cdot d\vec{S} = 0$$

Since both fields originate with currents div$\vec{B} = 0$ is still true. What about Ampere's Law?

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

but $\vec{j}$ now consists of free currents $\vec{j}_f$ and induced Amperian currents $\vec{j}_m$:

$$\begin{aligned}
\vec{\nabla} \times \vec{B} &= \mu_0 \vec{j}_f + \mu_0 \vec{j}_m \\
&= \mu_0 \vec{j}_f + \mu_o \vec{\nabla} \times \vec{M} \\
\rightarrow \vec{\nabla} \times (\vec{B} - \mu_0 \vec{M}) &= \mu_0 \vec{j}_f \\
\vec{\nabla} \times (\frac{1}{\mu_0} \vec{B} - \vec{M}) &= \vec{j}_f
\end{aligned}$$

We call $\quad \vec{B}/\mu_0 - \vec{M} = \vec{H} \quad$ the magnetic field strength. Its units are A/m.

Thus Ampere's Law becomes

$$\vec{\nabla} \times \vec{H} = \vec{j}_f$$

Or in integral form

$$\oint_L \vec{H} d\vec{l} = \int_S \vec{j}_f d\vec{S}$$

Again we have found a way of writing our fundamental laws in such a way that the polarisation and magnetisation effects of the medium are already accounted for.

We did relate $\vec{E}$, $\vec{D}$ and $\vec{P}$ via the electric susceptibility

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E}(1 + \chi_e)$$

For magnetic media we can define a magnetic susceptibility χ_m via $\vec{M} = \chi_m \vec{H}$

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} = \frac{1}{\mu_0}\vec{B} - \chi_m \vec{H}$$

$$\rightarrow \vec{B} = \mu_0(1 + \chi_m)\vec{H} = \mu_0 \mu_r \vec{H}$$

$\mu_r = (1 + \chi_m)$ defines an analogous material constant like ϵ_r. But there are differences. We saw $\epsilon_r \geq 1$ because the internal dipoles were exclusively created by external fields.

In magnetic materials each elementary particle, atom, molecule etc can have an intrinsic magnetic moment in addition to the induced ones.

4.13: Diamagnetism

If the material contains no magnetic dipole moments it is diamagnetic. All magnetic response is induced by the external field.

$$\chi_m < 0 \qquad\qquad |\chi_m| << 1$$

Typical values are:

$$\begin{aligned} \text{Bi:} \quad \chi_m &= -1.4 \times 10^{-5} \\ \text{Water:} \quad \chi_m &= -0.72 \times 10^{-6} \end{aligned}$$

4.14: Paramagnetism

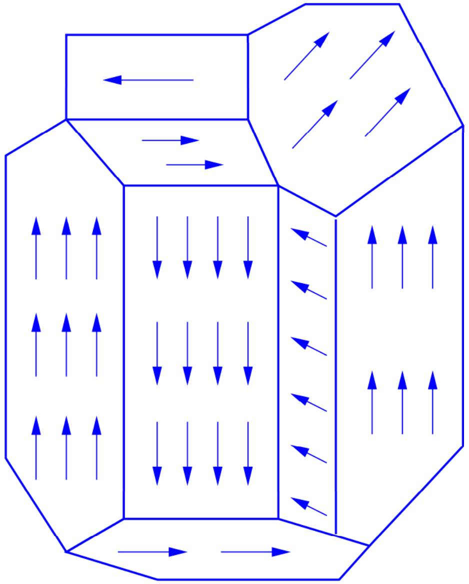
The material contains intrinsic magnetic dipole moments which get aligned by the external field

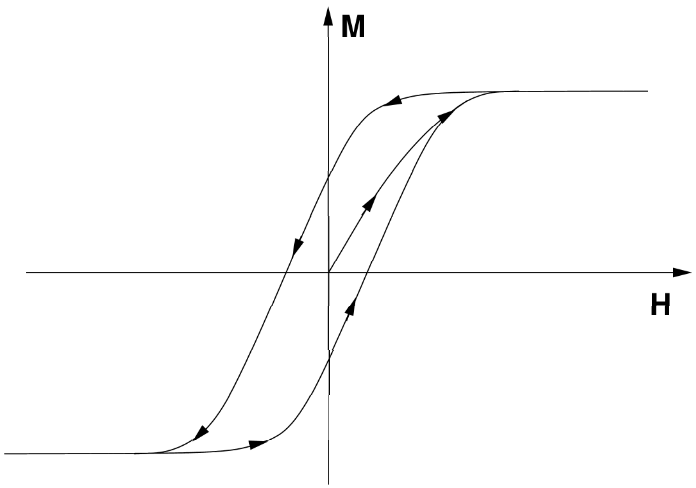
$$\chi_m > 0 \qquad\qquad |\chi_m| << 1$$

Typical values are:

$$\begin{aligned} \text{Platinum:} \quad \chi_m &= 1.93 \times 10^{-5} \\ \text{Liquid Oxigen:} \quad \chi_m &= 3.6 \times 10^{-4} \end{aligned}$$

4.15: Ferromagnetism





The magnetic moments of the material are so large that they align each other into magnetic domains. The magnetization of each domain is completely saturated.

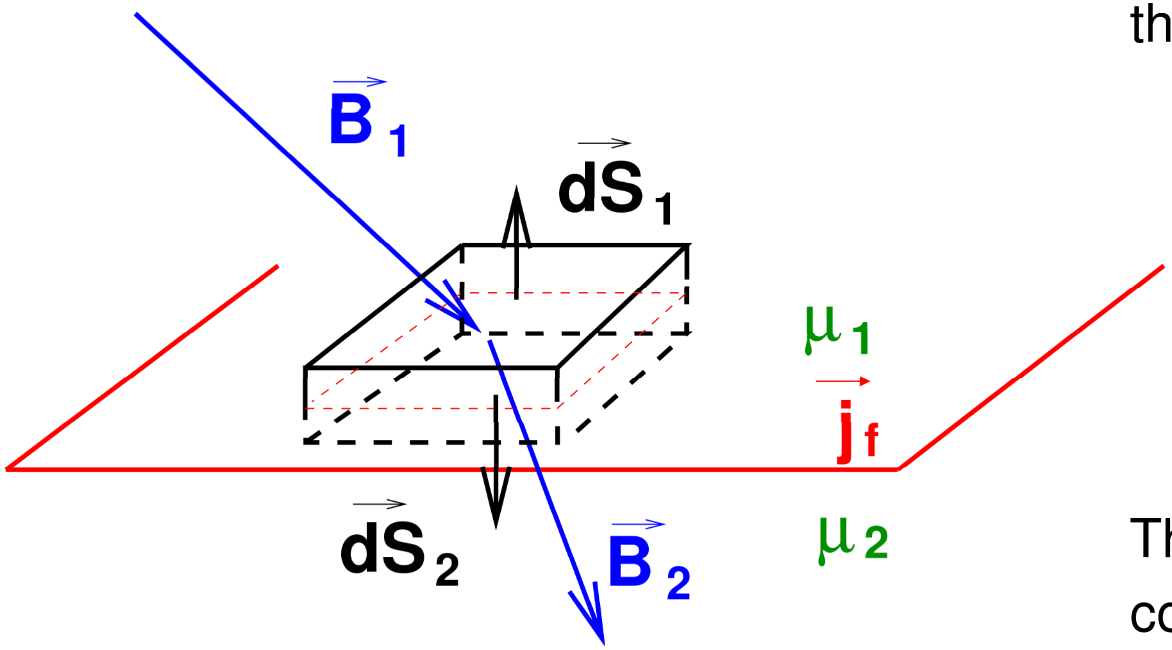
We can no longer use χ_m as a single valued constant, it even depends on the history!

This behaviour is also temperature dependent. Above a critical temperature (the Curie-Temperature) the material becomes paramagnetic. e.g.

$$\begin{aligned} \text{Fe:} \quad T_c &= 774^\circ C \\ \text{Co:} \quad T_c &= 1131^\circ C \end{aligned}$$

4.16: Continuity of $\vec{B}$

What are the continuity relations at the boundaries between materials with different μ_r?



We can use $\oint_S \vec{B} d\vec{S} = 0$ to argue that $B_{1\perp} = B_{2\perp}$:

$$0 = \oint \vec{B} d\vec{S} \;=\; \vec{B}_1 d\vec{S}_1 + \vec{B}_2 d\vec{S}_2$$

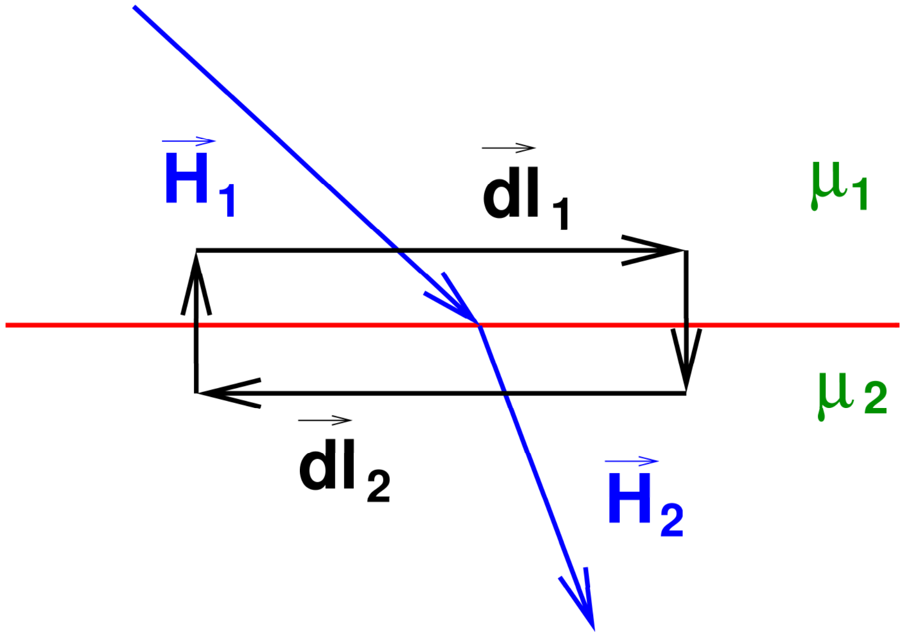
$$0 \;=\; B_{1\perp} - B_{2\perp}$$

$$B_{1\perp} \;=\; B_{2\perp}$$

The normal component of $\vec{B}$ is continuous across any boundary.

4.17: Continuity of $\vec{H}$

The continuity of $\vec{H}$ at a boundary is a bit more complicated. Again we consider a boundary between two materials with permeabilities μ_{r1} and μ_{r2}. The boundary layer may contain a surface current density.

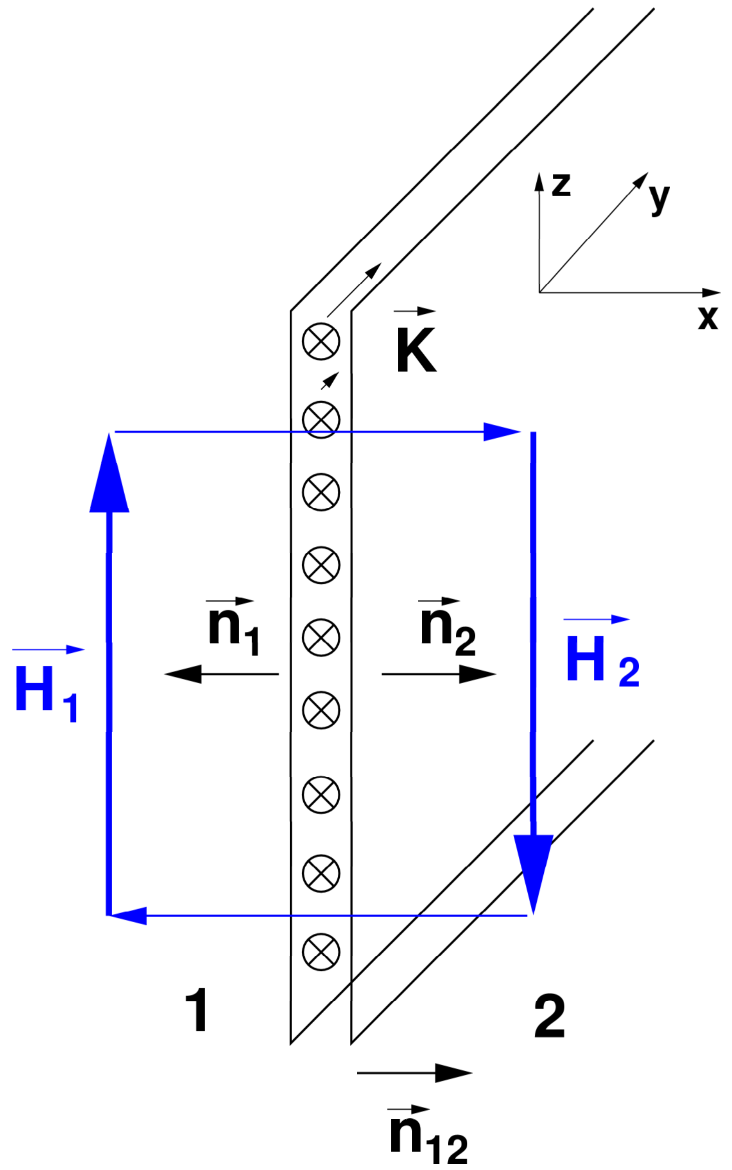


Case I: The boundary layer contains no currents:

$$
\begin{aligned}
\oint \vec{H}\vec{dl} = 0 &= \vec{H}_1 \cdot \vec{dl}_1 + \vec{H}_2 \cdot \vec{dl}_2 \\
0 &= H_{1||}dl - H_{2||}dl \\
\Rightarrow H_{1||} &= H_{2||}
\end{aligned}
$$

The tangential component of $\vec{H}$ is continuous across the boundary.

Case II: Now we consider a boundary that carries a current sheet.

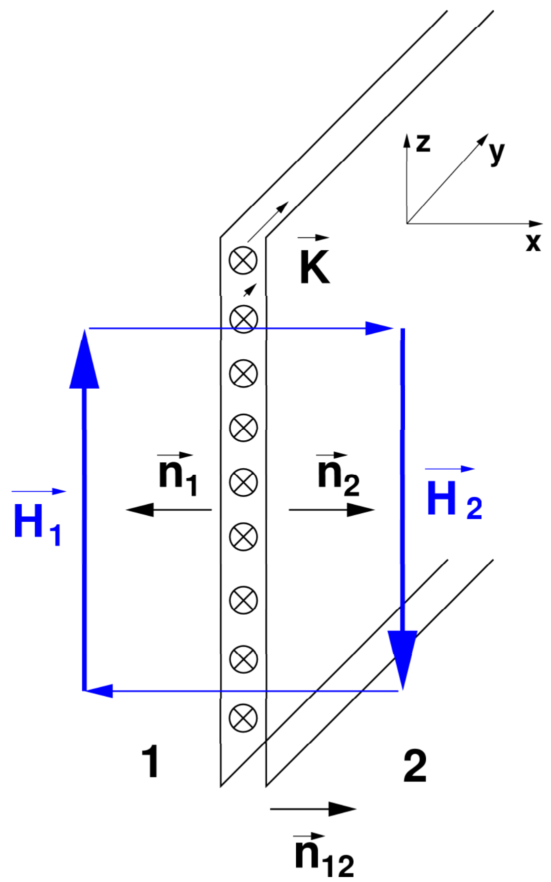


First we calculate the magnetic field created by that current sheet. We call the surface current density $\vec{K}$. From Ampere's law we can calculate $\vec{H}$ on either side:

$$H \cdot d + H \cdot d = K \cdot d \qquad \Rightarrow \qquad H = K/2$$

This can easily be generalised to a current sheet $\vec{K}$ in a plane with normal $\vec{n}$ on the side where we want to find $\vec{H}$:

$$\vec{H} = \frac{1}{2}\vec{K} \times \vec{n}$$



Now we split the magnetic field $\vec{H}$ up into a part generated by the current sheet $\vec{H}_K$ and one generated by all other currents $\vec{H}_O$:

$$\vec{H} = \vec{H}_K + \vec{H}_O$$

$\vec{H}_O$ is continuous at the boundary. $\vec{H}_K$ jumps from $\vec{H}_{K1} = (1/2)\vec{K} \times \vec{n}_1$ to $\vec{H}_{K2} = (1/2)\vec{K} \times \vec{n}_2$:

$$(\vec{H}_{K1} - \vec{H}_{K2}) \times \vec{n}_{12} = \vec{K}$$

If we add them back together we find

$$(\vec{H}_1 - \vec{H}_2) \times \vec{n}_{12} = \vec{K}$$

this means that the tangential component of $\vec{H}$ is discontinuous by $\vec{K}$.

4.18: Summary

We call $\vec{B}/\mu_0 - \vec{M} = \vec{H}$ the magnetic field strength. Its units are A/m^2.

Ampere's law is

$$\vec{\nabla} \times \vec{H} = \vec{j}_f \qquad \oint_L \vec{H}d\vec{l} = \int_S \vec{j}_f d\vec{S}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \oint_S \vec{B}d\vec{S} = 0$$

Diamagnetic: χ_m small and negative. Paramagnetic: χ_m small and positive.
Ferromagnetic: χ_m large, multivalued and dependent on the history.

The normal component of $\vec{B}$ is continuous across any boundary.

The tangential component of $\vec{H}$ is discontinuous by $\vec{K}$.

4.19: Electromagnetism

We will now leave static situations behind. From now on electric and magnetic fields, currents, charges and potentials will be allowed to vary with time and we will deduce the laws that govern their behaviour in the most general cases.

Michael Faraday noticed in 1831 noticed these experimental facts:

- Two coils can mutually induce a current in the other if the current in one is varied.

- A stationary coil could show a current if the magnetic flux through it changed.

- A moving coil could also show a current if it moved in a non-homogeneous magnetic field.

- The current in the coil is proportional to the conductance of the material
$\rightarrow$ An EMF is induced and the current follows from Ohm's law.

We start with a conducting wire moving through a homogeneous magnetic field:

The electrons in the wire experience a Lorentz force

$$\vec{F}_L = q\vec{v} \times \vec{B}$$

The charges move as if they are under the influence of an electric field

$$\vec{E} = \frac{1}{q}\vec{F} = \vec{v} \times \vec{B}$$

The Electromotance is given by

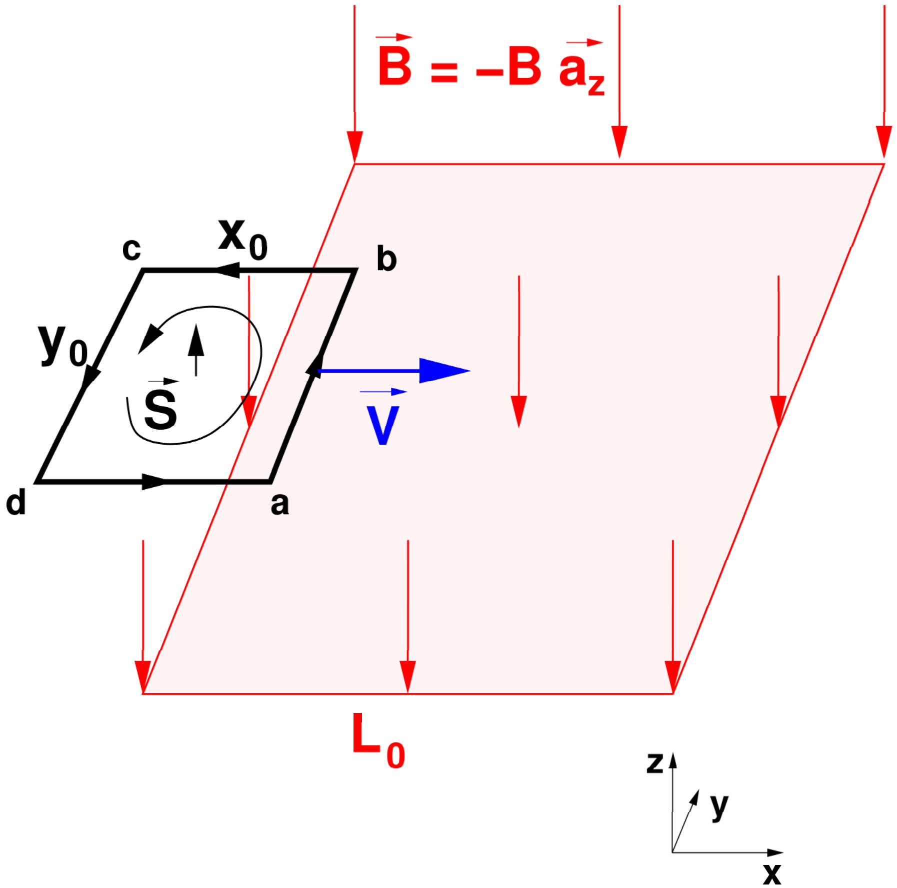
$$\mathcal{E} = \int_a^b \vec{E}\, d\vec{L} \qquad \text{or} \qquad d\mathcal{E} = \vec{E}\, d\vec{L}$$

Thus an circuit element of length $d\vec{L}$ moving with velocity $\vec{v}$ through a magnetic field $\vec{B}$ obtains an electromotance

$$d\mathcal{E} = \vec{E}\, d\vec{L} = (\vec{v} \times \vec{B}) d\vec{L}$$

And, if we have a closed circuit:

$$\mathcal{E} = \oint_L (\vec{v} \times \vec{B})\, d\vec{L}$$



We now look at an entire circuit to determine all directions uniquely.

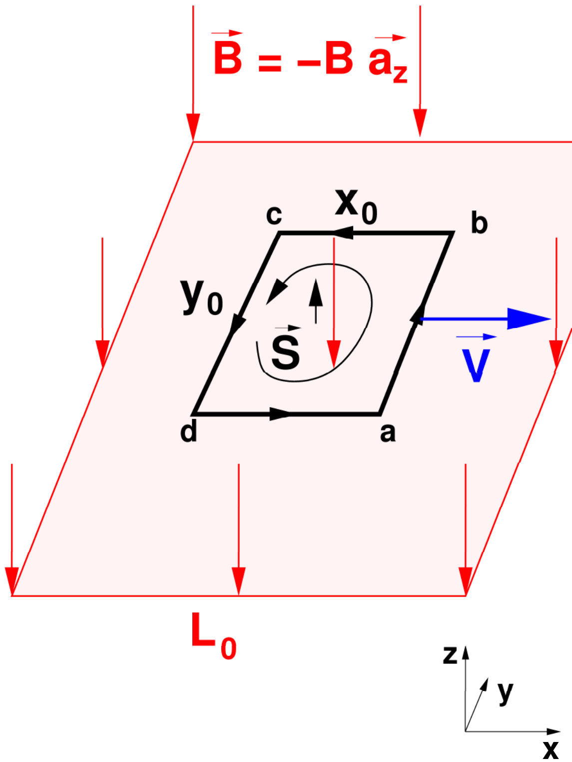
A rectangular circuit enters a region containing a homogeneous magnetic field:

$$\vec{S} = x_0\, y_0\, \underline{\hat{\mathsf{a}}}x \times \underline{\hat{\mathsf{a}}}y = x_0\, y_0\, \underline{\hat{\mathsf{a}}}z$$

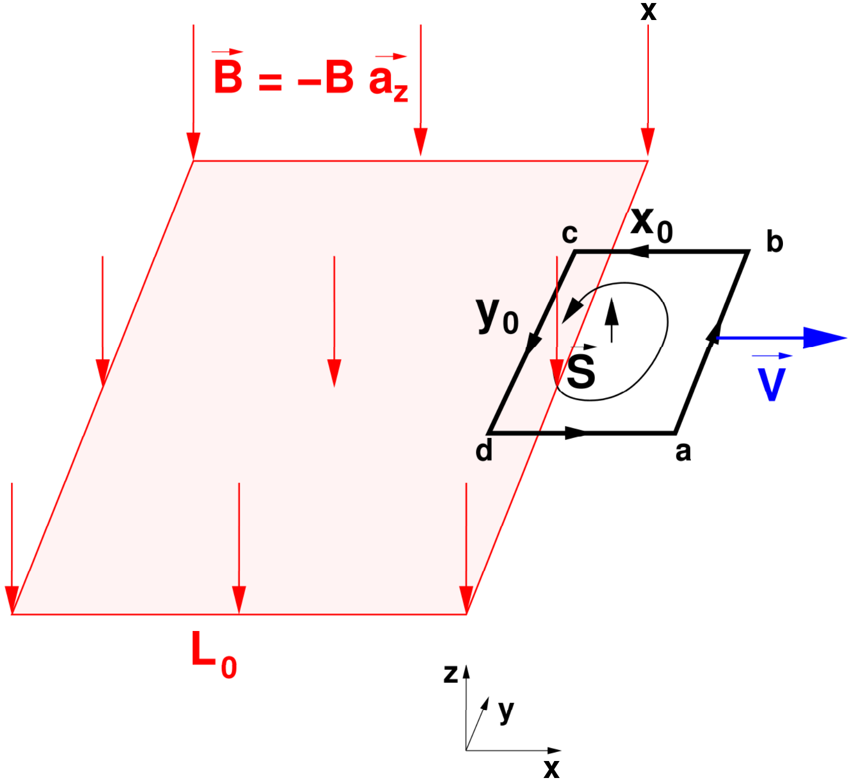
The $\mathcal{E}$ in piece $a \to b$ is

$$\begin{aligned} \mathcal{E} &= \int_a^b \vec{E}\, d\vec{L} = \int_a^b (\vec{v} \times \vec{B})\, d\vec{L} \\ &= \int_a^b (v\, \underline{\hat{\mathsf{a}}}x \times B(-\underline{\hat{\mathsf{a}}}z)) \cdot dy\, \hat{a}_y = v\, B\, y_0 \end{aligned}$$

$\mathcal{E}$ in $b \to c \to d \to a$ is zero.

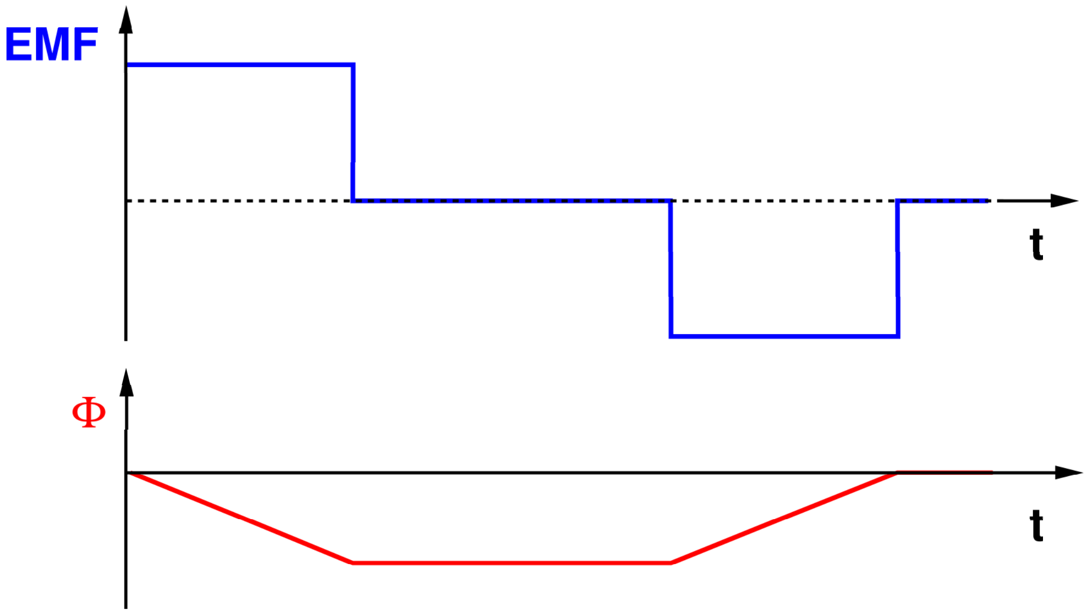


As the rectangle is fully in the field, the contributions $a \to b$ and $d \to c$ are equal and opposite: no current flows.



As the rectangle leaves, we only have a contribution from $c \to d$:

$$\mathcal{E}_{c\to d} = -\mathcal{E}_{a\to b}$$



The EMF is:

$$\mathcal{E} = -k\frac{d\Phi}{dt}$$

and in SI units $k = 1$

This is Faraday's Law:

$$\oint \vec{E}d\vec{L} = -\frac{d}{dt}\int_S \vec{B}d\vec{S}$$

4.20: Lenz' law

To find the direction of the induced current we can make use of Lenz' Law:

"The voltage (EMF) induced by a changing flux has a polarity such that the current established in a closed path gives rise to a flux which opposes *the change in flux."*

Lenz 1845

In the example above that means:

- The current must be counterclockwise as we go into the field
  to produce $\vec{B}$ *opposing* external $\vec{B}$.

- The current will be clockwise as we leave the field
  to produce $\vec{B}$ *reinforcing* external $\vec{B}$.

4.21: Example

A circular wire loop in the $x - y$ - plane grows in radius:

$$R(t) = R_0 \cdot \sqrt{\frac{t}{\tau}}$$

It sits in a time independent magnetic field $\vec{B} = B_0\,\hat{a}_z$. What are the induced EMF and the induced current?

The flux is $\Phi(t) = \vec{B} \cdot \vec{A}(t) = \pi\, B\, R_0^2\, t/\tau$

If we choose the orientation of $\vec{A} = A\vec{a}_z$ the loop has a positive orientation counter-clockwise. (right hand rule)

$$\int \vec{E}\, d\vec{L} = -\frac{d\Phi}{dt} = \frac{-\pi\, B\, R_0^2}{\tau}$$

The - sign indicates that the EMF is *clockwise*.

Thus we get a clockwise current, which generates a magnetic field *down*
(This opposes the change in flux)

4.22: Faraday's law in differential form

Faraday's Law in integral form is

$$\oint_L \vec{E}\, d\vec{L} = -\frac{d}{dt}\int_S \vec{B} d\vec{S}$$

In differential form we get

$$\oint \vec{E}\, d\vec{L} = \int_S \text{curl}\vec{E}\, d\vec{S} = -\frac{d}{dt}\int_S \vec{B}\, d\vec{S}$$

How do we treat the time derivative? The area may change with time !?!

But $\vec{E}$ is the field at the position of $d\vec{L}$, regardless of any motion of $d\vec{L}$.

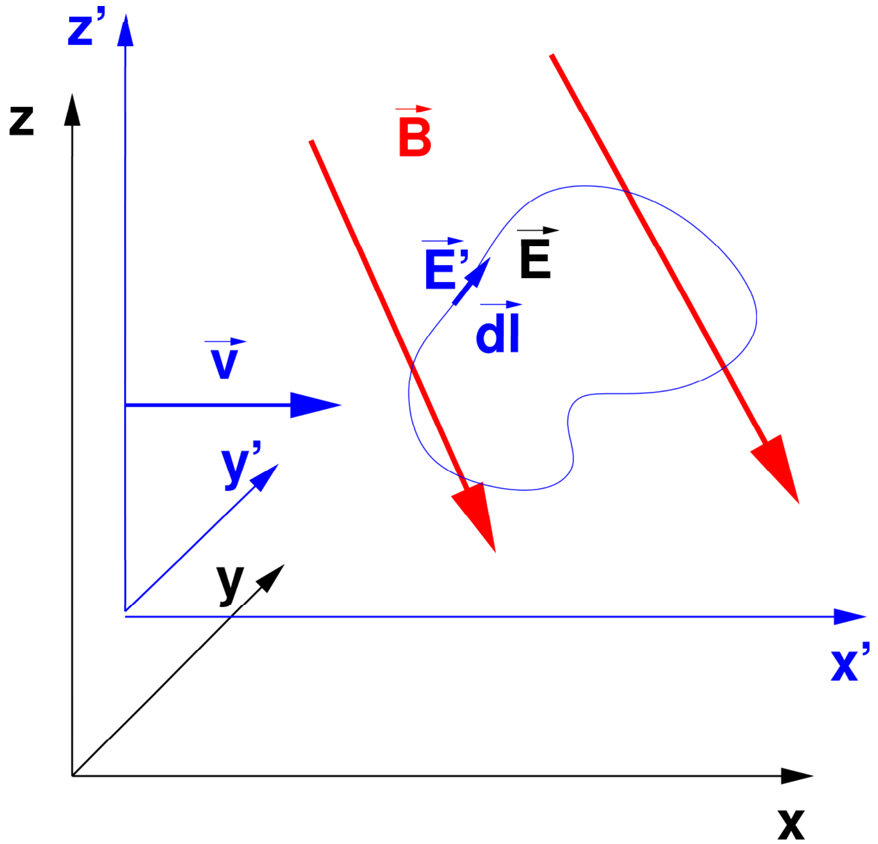
Thus if we want to have $\vec{E}$ and $\vec{B}$ measured in the same frame of reference, we choose that frame in which the loop is stationary, but now $\vec{B}$ may be moving. Then

$$\begin{aligned}
\oint_L \vec{E}\, d\vec{L} &= \int_S \text{curl}\vec{E}\, d\vec{S} \\
&= -\frac{d}{dt}\int_S \vec{B}\, d\vec{S} \\
&= \int_S -\frac{\partial \vec{B}}{\partial t}\, d\vec{S}
\end{aligned}$$

and since again the two integrals must be equal for any arbitrary surface S, the integrands must be identical:

$$\text{curl}\, \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This is Faraday's Law in differential form.



What if we allow $d\vec{L}$ to move with velocity $\vec{v}$?
We then had a contribution to the EMF of

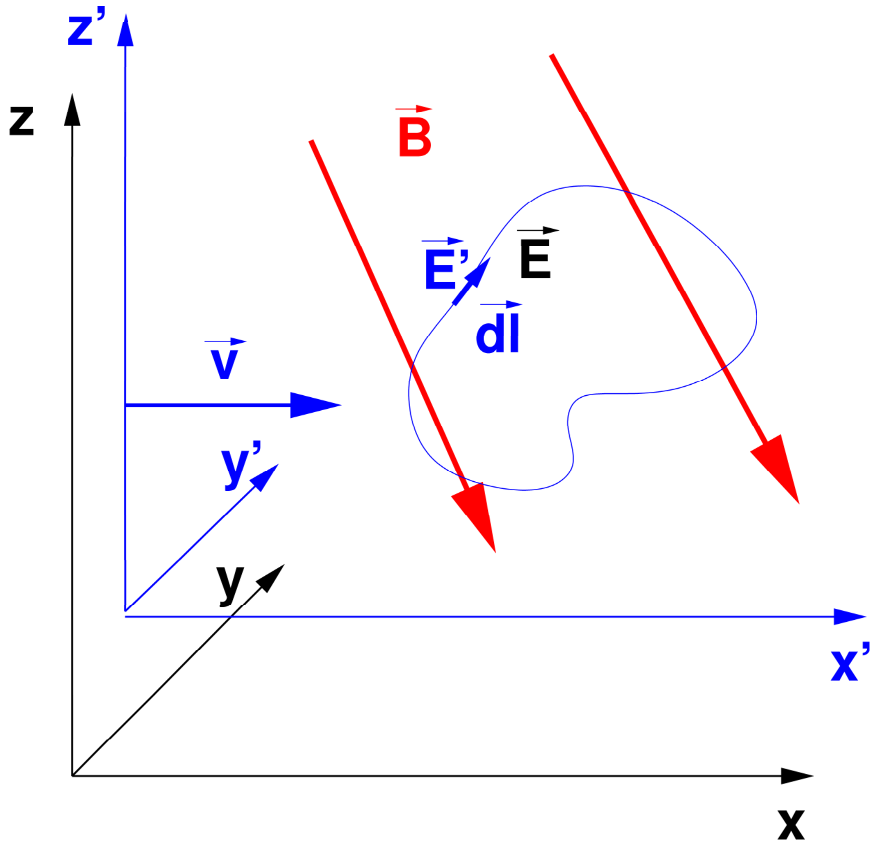
$$\mathcal{E} = \int \vec{E} d\vec{L} = \int (\vec{v} \times \vec{B}) d\vec{L}$$

If we add both we get

$$\int \vec{E}\, d\vec{L} = \int \vec{v} \times \vec{B}\, d\vec{L} - \frac{d}{dt} \int \vec{B} \cdot d\vec{S}$$

Both $\vec{E}$ and $\vec{B}$ are measured in the laboratory frame of reference where $d\vec{L}$ is moving.

$$\int (\vec{E} - \vec{v} \times \vec{B}) d\vec{L} = -\frac{d}{dt} \int \vec{B}\, d\vec{S}$$



In the frame of reference where $d\vec{L}$ is fixed we have

$$\int \vec{E}'\, d\vec{L} = -\frac{d}{dt}\int \vec{B}\, d\vec{S}$$

So we get a relationship between $\vec{E}'$ in the moving frame and $\vec{E}$ in the laboratory frame:

$$\vec{E}' = \vec{E} - \vec{v} \times \vec{B}$$

if the moving frame moves with velocity $\vec{v}$ with respect to the laboratory frame.

An observer may see a electric of magnetic field or a combination of both depending on his state of motion!

4.23: Example

$$\vec{E} = E_0(-y, x, 0) \qquad \text{find} \quad \vec{B}:$$

$$\text{curl}\vec{E} = (0, 0, 2E_0) = -\frac{\partial \vec{B}}{\partial t}$$

$$\frac{\partial B_{x,y}}{\partial t} = 0 \rightarrow B_{x,y} = B^0_{x,y} = const$$

$$\frac{\partial B_z}{\partial t} = -2\,E_0$$

$$\rightarrow B_z = -2E_0 t + B^0_z$$

Do the units work out?

$[\vec{E}]$ = V/m, $[x, y]$ = m $\rightarrow$ $[E_0]$ =V/m^2.
$[\vec{B}]$ = T = V s/m^2 and $[(\partial B/\partial t)]$ = V/m^2

4.24: Summary

An circuit element of length $d\vec{L}$ moving with velocity $\vec{v}$ through a magnetic field $\vec{B}$ obtains an electromotance

$$d\mathcal{E} = \vec{E}\,d\vec{L} = (\vec{v} \times \vec{B})d\vec{L}$$

Faraday's Law in integral form is:

$$\oint \vec{E}d\vec{L} = -\frac{d}{dt}\int_S \vec{B}d\vec{S}$$

Lenz' Law:

"The voltage (EMF) induced by a changing flux has a polarity such that the current established in a closed path gives rise to a flux which opposes *the change in flux."*

Faraday’s Law in differential form:

$$\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The relationship between $\vec{E}'$ in a frame moving with velovity $\vec{v}$ and $\vec{E}$ in the laboratory frame:

$$\vec{E}' = \vec{E} - \vec{v} \times \vec{B}$$

4.25: Ampere's Law

We have obtained equations that relate the electric and magnetic fields to currents, charges and each other. So far we have

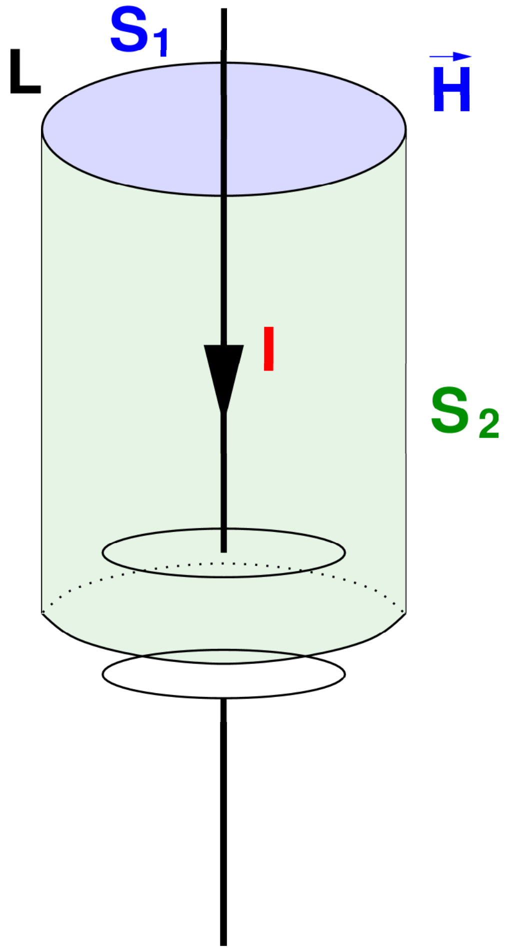
$$\oint_S \vec{B}\, d\vec{S} = 0 \qquad\qquad \text{div}\vec{B} = 0$$

$$\oint_L \vec{E}\, d\vec{L} = -\frac{d}{dt}\oint_S \vec{B}\, d\vec{S} \qquad\qquad \text{curl}\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\oint_S \vec{D}\, d\vec{S} = \int_V \rho\, dV \qquad\qquad \text{div}\vec{D} = \rho$$

$$\oint_L \vec{H}\, d\vec{L} = \int_S \vec{j}\, d\vec{S} \qquad\qquad \text{curl}\vec{H} = \vec{j}$$

If we examine these equations they tell us everything we may need to know *exept* in the case of time-varying electric fields. That will be the last addition we need to make and it will modify Ampere's Law.



Consider a straight wire that charges a capacitor. We calculate the magnetic field strength $\vec{H}$ via the surface S_1 shown in the figure. Clearly $\vec{H} \neq 0$, and more precisely

$$\oint_L \vec{H}\, d\vec{L} = \int_{S_1} \vec{j}\, d\vec{S} = I$$

But that surface is only one possible surface. What happens if we choose surface S_2 as a "tophat" that passes between the plates of the capacitor? No current passes through S_2, so

$$\oint_L \vec{H}\, d\vec{L} = \int_{S_2} \vec{j}\, d\vec{S} = 0 \qquad ?$$

Clearly, there must be a non-zero, unique field $\vec{H}$, because we could get it from the Biot-Savart Law!

And another inconsistency rears its head.

$$\text{curl}\vec{H} = \vec{j}$$

Take the diverence on both sides

$$\text{div curl}\vec{H} = \text{div}\vec{j}$$

The left hand side is always zero. The right hand side we remember from the continuity equation:

$$\text{div}\vec{j} = -\frac{\partial \rho}{\partial t}$$

and is *not* zero! We clearly need to extend Ampere's Law, and the continuity equation tells us how.

One of Maxwell's equations was

$$\text{div}\vec{D} = \rho$$

$$\Rightarrow \quad \text{div}\vec{j} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}\text{div}\vec{D} = -\text{div}\,\frac{\partial \vec{D}}{\partial t}$$

$$\Rightarrow \quad \text{div}\,(\vec{j} + \frac{\partial \vec{D}}{\partial t}) = 0$$

This suggests an extension to Ampere's Law in the following way:

$$\text{curl}\,\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

Let's analyse the solution to the "two surface paradox" now:

A constant voltage V_0 is applied and charges the capacitor via a long wire with resistance R. At $t = 0$ the charge on the capacitor is zero

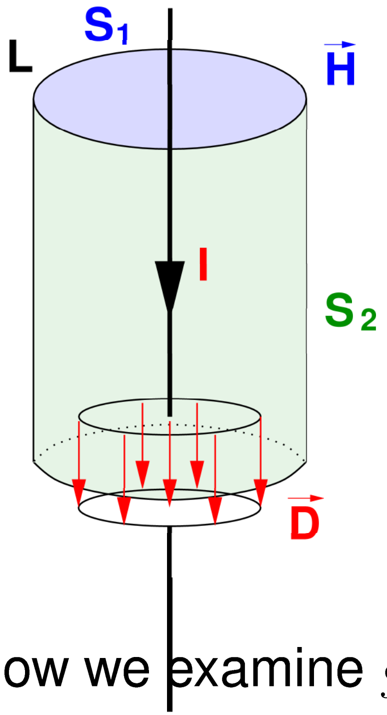
$$Q(t = 0) = 0$$

The charge as a function of time is given by

$$Q(t) = Q_0(1 - \exp(-t/RC)) = V_0 C(1 - \exp(-t/RC))$$

The current thus becomes

$$I = \frac{dQ}{dt} = \frac{Q_0}{RC}\exp(-t/RC) = \frac{V_0}{R}\exp(-t/RC)$$



The $\vec{D}$ field between the plates of area A is

$$|\vec{D}(t)| = \frac{Q(t)}{A} \quad \Rightarrow \quad \frac{\partial D}{\partial t} = \frac{1}{A}\frac{\partial Q(t)}{\partial t} = \frac{Q_0}{ARC}\,\mathrm{exp}(-t/RC)$$

$$\frac{\partial D}{\partial t} = \frac{V_0}{AR}\,\mathrm{exp}(-t/RC)$$

Now we examine $\oint_L \vec{H}\,d\vec{L}$: First S_1:

$$\oint_L \vec{H}\,d\vec{L} = \int_{S_1} (\vec{j} + \frac{\partial \vec{D}}{\partial t})\,d\vec{S} = \int_{S_1} \vec{j}\,d\vec{S} = I = \frac{V_0\,\mathrm{exp}(-t/RC)}{R}$$

Now S_2:

$$\oint_L \vec{H}\,d\vec{L} = \int_{S_2} (\vec{j} + \frac{\partial \vec{D}}{\partial t})d\vec{S} = \int_{S_2} \frac{\partial \vec{D}}{\partial t} d\vec{S} = \frac{V_0}{AR}\,\mathrm{exp}(-t/RC) \cdot A = \frac{V_0\,\mathrm{exp}(-t/RC)}{R}$$

And our paradox is resolved.

4.26: Maxwell's Equations

We now have a full set of Maxwell equations:

$$\oint_S \vec{B}\, d\vec{S} = 0 \qquad \mathrm{div}\vec{B} = 0$$

$$\oint_S \vec{D}\, d\vec{S} = \int_V \rho\, dV \qquad \mathrm{div}\vec{D} = \rho$$

$$\oint_L \vec{E}\, dL = -\frac{d}{dt}\oint_S \vec{B}\, d\vec{S} \qquad \mathrm{curl}\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\oint_L \vec{H}\, dL = \int_S (\vec{j} + \frac{\partial \vec{D}}{\partial t})\, d\vec{S} \qquad \mathrm{curl}\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

Together with

$$\vec{F}_L = q(\vec{E} + \vec{v} \times \vec{B}) \qquad \vec{F}_C = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{\vec{r}}{|\vec{r}|^3}$$

$$d\vec{B} = \frac{I\,\mu_0}{4\pi}\frac{d\vec{L} \times \vec{r}}{|\vec{r}|^3}$$

$$\vec{j} = \sigma\vec{E} \qquad \text{div}\vec{j} = -\frac{\partial\rho}{\partial t}$$

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} \qquad \vec{P} = \chi_e\vec{E}$$

$$\vec{D} = \epsilon_0\epsilon_r\vec{E} \qquad \text{(in LIH media)}$$

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \qquad \vec{M} = \chi_m\vec{H}$$

$$\vec{B} = \mu_0\mu_r\vec{H} \qquad \text{(in LIH diamagnetic/paramagnetic media)}$$

This is all we need to understand in Electromagnetism!

4.27: Maxwell's Equations

What do they mean?

$$\text{div}\vec{B} = 0$$

Magnetic field lines close in on themselves, there are no magnetic monopoles as sources of the magnetic field.

$$\text{div}\vec{D} = \rho$$

Electric charges are the sources of the electric field.

$$\text{curl}\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The vorticity of the electric field is caused by time-varying magnetic fields.

$$\text{curl}\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

The vorticity of the magnetic field is caused by currents and time-varying electric fields.

4.28: The many faces of Maxwell's equations

The general form of Maxwell's equations is

$$\text{div}\vec{D} = \rho \qquad \text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{div}\vec{B} = 0 \qquad \text{curl}\,\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

But in many cases simpler, specialised versions are useful to memorise.

In a first step assume that any media present are linear, isotropic and homogeneous (LIH). Then

$$\vec{D} = \epsilon_0 \epsilon_r \vec{E} \qquad \text{and} \qquad \vec{B} = \mu_0 \mu_r \vec{H}$$

We can then write Maxwell's equations using just $\vec{E}$ and $\vec{B}$:

$$\text{div}\vec{E} = \frac{\rho}{\epsilon_0 \epsilon_r} \qquad \text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{div}\vec{B} = 0 \qquad \text{curl}\,\vec{B} = \mu_0 \mu_r \epsilon_0 \epsilon_r \frac{\partial \vec{E}}{\partial t} + \mu_0 \mu_r \vec{j}$$

Exercise: find the corresponding integral forms using Stokes' and Gauss' Laws.

A common abstraction used is to neglect any influence of media, but still allow the presence of charges and currents. Then $\epsilon_r = 1 = \mu_r$ and we get Maxwell's equations in free space with charges and currents:

$$\text{div}\vec{E} = \frac{\rho}{\epsilon_0} \qquad \text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{div}\vec{B} = 0 \qquad \text{curl}\,\vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j}$$

Since there is no matter around the use of $\vec{B}$ and $\vec{E}$ only is preferred.

Again, transform these into integral forms.

Finally we write down Maxwell's equations in vacuum, with no charges or currents, just fields:

$$\begin{aligned} \text{div}\vec{E} &= 0 \qquad \text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \text{div}\vec{B} &= 0 \qquad \text{curl}\,\vec{B} = \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} \end{aligned}$$

This last form is amazing: We started our discussions of electric and magnetic fields starting from charges and currents. Yet, even in the absence of both, then fields can take on a life of their own. We don't even require a medium: Vacuum itself sustains time-dependent electromagnetic fields. This is light.

4.29: Light

We know that any wave obeys the wave equation

$$\frac{\partial^2 \vec{F}}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \vec{F}}{\partial t^2} \qquad \text{with velocity } v \text{ and propagation in x-direction}$$

If we allow propagation in a general direction $\vec{r} = (x, y, z)$ the left hand side gets expanded to

$$\frac{\partial^2 \vec{F}}{\partial x^2} + \frac{\partial^2 \vec{F}}{\partial y^2} + \frac{\partial^2 \vec{F}}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \vec{F}}{\partial t^2}$$

Or, using the Laplace operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\nabla^2 \vec{F} = \frac{1}{v^2}\frac{\partial^2 \vec{F}}{\partial t^2}$$

If we can transform our Maxwell equations into this form, then we know we are dealing with a wave moving at speed v.

We start with

$$\begin{aligned}\text{curl}\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \text{curl curl}\vec{E} &= \text{curl}(-\frac{\partial \vec{B}}{\partial t}) = -\frac{\partial}{\partial t}\text{curl}\vec{B}\end{aligned}$$

Now substitute $\text{curl}\vec{B} = \mu_0\epsilon_0(\partial\vec{E})/(\partial t)$:

$$\text{curl curl}\vec{E} = -\frac{\partial}{\partial t}\mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} = -\mu_0\epsilon_0\frac{\partial^2 \vec{E}}{\partial t^2}$$

The lefthand side can be rewritten:

$$\begin{aligned}\text{curl curl}\vec{E} &= \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{E} \\ &= \text{grad}(\text{div}\vec{E}) - \nabla^2\vec{E}\end{aligned}$$

But in vacuum div$\vec{E} = 0$, so

$$-\nabla^2 \vec{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

or

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

A wave equation for the electric field!

What about $\vec{B}$?

$$\begin{aligned}
\text{curl}\vec{B} &= \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \\
\text{curl curl}\vec{B} &= \mu_0\epsilon_0\frac{\partial}{\partial t}\text{curl}\vec{E} \\
\text{grad}(\text{div}\vec{B}) - \nabla^2\vec{B} &= \mu_0\epsilon_0\frac{\partial}{\partial t}(-\frac{\partial\vec{B}}{\partial t})
\end{aligned}$$

But div$\vec{B} = 0$, Thus

$$\nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

as well.

So, the electric and magnetic fields both obey a wave equation in free space without charges or currents. Does that mean we have “electric light” and “magnetic light”? Examine the propagation more closely.

We start with a time-varying electric field. This induces a time-varying magnetic field via curl$\vec{B} = \mu_0\epsilon_0(\partial\vec{E})/(\partial t)$.

This magnetic field in turn induces a time-varying electric field via curl$\vec{E} = -(\partial\vec{B})/(\partial t)$, which in turn induces a time-varying magnetic field which in turn induces a time-varying electric field etc.

Thus the electric and magnetic fields cannot exist alone. Any electric wave is always accompanied by a magnetic wave and vice versa.

This process of cross-excitation is what allows electromagnetic waves to propagate through vacuum without needing any medium at all.

At what speed do these waves travel? By comparing

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

with out general wave equation

$$\nabla^2 \vec{F} = \frac{1}{v^2} \frac{\partial^2 \vec{F}}{\partial t^2}$$

we can read off

$$v^2 = \frac{1}{\mu_0 \epsilon_0} \qquad \text{or} \qquad v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = \frac{1}{\sqrt{8.854 \times 10^{-12} \cdot \frac{As}{Vm} \cdot 4\pi \times 10^{-7} \frac{Vs}{Am}}}$$

$\rightarrow v = 299792458\ m/s = c$

This is the speed of light in vacuum.

4.30: Worked example

An electric field is given by $\vec{E}(\vec{r},t) = E_0\ \underline{\hat{\mathsf{a}}}_y \cdot \mathrm{sin}k(x-ct)$

- Show that it fulfills the wave equation

- What is the corresponding magnetic field $\vec{B}(\vec{r},t)$?

- Show that $\vec{B}(\vec{r},t)$ also fulfills the wave equation

$$
\begin{aligned}
\nabla^2 \vec{E} &= \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} \\
&= \frac{\partial^2}{\partial x^2} E_0 \, \underline{\hat{\mathsf{a}}}_y \sin k(x - ct) \\
&= -E_0 k^2 \, \underline{\hat{\mathsf{a}}}_y \sin k(x - ct) \\
&= -k^2 \vec{E}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \vec{E}}{\partial t^2} &= \frac{\partial}{\partial t} \left[-kcE_0 \, \underline{\hat{\mathsf{a}}}_y \cos k(x - ct) \right] \\
&= -E_0 k^2 c^2 \, \underline{\hat{\mathsf{a}}}_y \sin k(x - ct) \\
&= -k^2 c^2 \vec{E}
\end{aligned}
$$

Now substiture these into the wave equation:

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$-k^2 \vec{E} = \frac{-k^2 c^2}{c^2}\vec{E}$$

$$\vec{E} = \vec{E}$$

This electric field fulfills the wave equation.

To find $\vec{B}$ we have to use Maxwell's equations. First we use

$$\text{curl}\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\begin{aligned}
\text{curl}\vec{E} &= \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \\
&= \left(0 - 0, 0 - 0, \frac{\partial E_0 \sin k(x - ct)}{\partial x} - 0\right) \\
&= (0, 0, kE_0 \cos k(x - ct)) = -\frac{\partial \vec{B}}{\partial t}
\end{aligned}$$

Thus we find B_x and B_y are constant in time.

B_z is found by integration:

$$\begin{aligned} B_z &= -\int k E_0 \cos k(x-ct) dt \\ &= \frac{-1}{-kc} E_0 k \sin k(x-ct) + const \\ &= +\frac{E_0}{c} \sin k(x-ct) + const \end{aligned}$$

$$\vec{B} = (0, 0, \frac{E_0}{c} \sin k(x-ct)) + \vec{C}$$

Now we check our result with

$$\text{curl}\vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

$$
\begin{aligned}
\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} &= \frac{1}{c^2}(-kc)E_0\,\underline{\hat{\mathsf{a}}}_y \cos k(x-ct) \\
&= \frac{-k}{c}E_0\,\underline{\hat{\mathsf{a}}}_y \cos k(x-ct) \\
\mathrm{curl}\vec{B} &= \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}, \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}, \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = \frac{-k}{c}E_0\,\underline{\hat{\mathsf{a}}}_y \cos k(x-ct)
\end{aligned}
$$

Now compare all three components individually:

$$x: \quad \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = 0 \qquad \text{A}$$

$$y: \quad \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \frac{-k}{c}E_0 \cos k(x-ct) \qquad \text{B}$$

$$z: \quad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0 \qquad \text{C}$$

A and C are obviously fulfilled. Check B:

$$-\frac{\partial B_z}{\partial x} = -\frac{\partial}{\partial x}\left[\frac{E_0}{c}\sin k(x-ct)\right] = \frac{-E_0 k}{c}\cos k(x-ct)$$

$$= -\frac{1}{c^2}\frac{\partial E_y}{\partial t}$$

$$\text{curl}\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

Since we are in vacuum no steady currents exist that could generate a constant $\vec{B}$. Thus we can set all constants of integration to zero:

$$\vec{B} = (0, 0, \frac{E_0}{c} \sin k(x - ct))$$

Lets examine directions. Both $\vec{E}$ and $\vec{B}$ propagate into positive x-direction. $\vec{E}$ is directed parallel to the y-axis and $\vec{B}$ is directed parallel to the z-axis.

$\vec{E}$ is perpendicular to $\vec{B}$

The direction of propagation is perpendicular to both $\vec{E}$ and $\vec{B}$. This is generally true:

Consider a general electric field

$$\vec{E} = \vec{E}_0 \sin(\vec{k} \cdot \vec{r} - wt)$$

propagating in direction $\vec{k}$.

$$\begin{aligned}\mathrm{div}\vec{E} = 0 &= \vec{E}_0 \left(\frac{\partial}{\partial x} \sin(\vec{k}\vec{r} - wt), \frac{\partial}{\partial y} \sin(\vec{k}\vec{r} - wt), \frac{\partial}{\partial z} \sin(\vec{k}\vec{r} - wt), \right) \\ 0 &= \vec{E}_0 \cdot \left(k_x \cos(\vec{k}\vec{r} - wt), k_y \cos(\vec{k}\vec{r} - wt), k_z \cos(\vec{k}\vec{r} - wt) \right) \\ 0 &= \vec{E}_0 \cdot \vec{k} \cdot \cos(\vec{k}\vec{r} - wt)\end{aligned}$$

$\rightarrow \vec{E}_0 \perp \vec{k}$

The same arguments shows $\vec{B} \perp \vec{k}$.

If $\vec{\hat{k}}$ is a unit vector in direction $\vec{k}$ we can write

$$\vec{B} = \frac{1}{c}(\vec{\hat{k}} \times \vec{E})$$

The magnitude of the magnetic field is a factor c smaller than the magnitude of the electric field.

E.g.: in the vicinity of a radio transmitter the electric field has a magnitude of 1 V/m. The corresponding magnetic field has a magnitude

$$\frac{1V/m}{3 \times 10^8 m/s} = 3 \times 10^{-9} \frac{V\, s}{m^2} = 3 \times 10^{-9} T$$

Compared to the earths magnetic field $10^{-4} T$ this is tiny.

4.31: Summary

Maxwell's equations predict electromagnetic waves propagating at the speed of light.

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

$\vec{B}$ and $\vec{E}$ are perpendicular to each other and the direction of propagation

The magnitude of $\vec{B}$ is a factor c smaller than the magnitude of $\vec{E}$.

The general form of Maxwell's equations is

$$\text{div}\vec{D} = \rho \qquad \text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{div}\vec{B} = 0 \qquad \text{curl}\,\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

Maxwell's equations in free space with charges and currents (but no condensed matter):

$$\text{div}\vec{E} = \frac{\rho}{\epsilon_0} \qquad \text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{div}\vec{B} = 0 \qquad \text{curl}\,\vec{B} = \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} + \mu_0\vec{j}$$

Maxwell's equations in vacuum (no charges or currents):

$$\text{div}\vec{E} = 0 \qquad \text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{div}\vec{B} = 0 \qquad \text{curl}\,\vec{B} = \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

4.32: Light in media

How does the propagation of light change inside LIH, non-conducting media?

Maxwell's equations in LIH media were:

$$\mathrm{div}\vec{E} = \frac{\rho}{\epsilon_0\epsilon_r} \qquad \mathrm{curl}\,\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\mathrm{div}\vec{B} = 0 \qquad \mathrm{curl}\,\vec{B} = \mu_0\mu_r\epsilon_0\epsilon_r\frac{\partial\vec{E}}{\partial t} + \mu_0\mu_r\vec{j}$$

In the absence of currents and charges (i.e. in a piece of glass) this becomes:

$$\mathrm{div}\vec{E} = 0 \qquad \mathrm{curl}\,\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\mathrm{div}\vec{B} = 0 \qquad \mathrm{curl}\,\vec{B} = \mu_0\mu_r\epsilon_0\epsilon_r\frac{\partial\vec{E}}{\partial t}$$

Compare with the vacuum form. The only difference is the occurrence of $\mu_0\mu_r\epsilon_0\epsilon_r$ instead of $\mu_0\epsilon_0$. The wave equations in matter are then

$$\nabla^2\vec{E} = \mu_0\mu_r\epsilon_0\epsilon_r\frac{\partial^2\vec{E}}{\partial t^2} \qquad \text{and} \qquad \nabla^2\vec{B} = \mu_0\mu_r\epsilon_0\epsilon_r\frac{\partial^2\vec{B}}{\partial t^2}$$

The speed of propagation is changed:

$$v = \frac{1}{\sqrt{\mu_0\mu_r\epsilon_0\epsilon_r}} = \frac{c}{\sqrt{\epsilon_r\mu_r}} = \frac{c}{n}$$

We call $n = \sqrt{\epsilon_r\mu_r}$ the index of refraction.

What happens if the wave encounters a conductor?

4.33: EM waves in conductors

Maxwell's equations in a conductor without net free charges can be rewritten using Ohm's Law $\vec{j} = \sigma \vec{E}$:

$$\mathrm{div}\vec{E} = 0 \qquad \mathrm{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\mathrm{div}\vec{B} = 0 \qquad \mathrm{curl}\,\vec{B} = \mu_0\mu_r\epsilon_0\epsilon_r\frac{\partial \vec{E}}{\partial t} + \mu_0\mu_r\sigma\vec{E}$$

This looks different. Can we get a wave equation from these?

$$\begin{aligned}
\mathrm{curl}\,\mathrm{curl}\,\vec{B} &= \mu_0\mu_r\epsilon_0\epsilon_r\frac{\partial}{\partial t}\mathrm{curl}\vec{E} + \mu_0\mu_r\sigma\mathrm{curl}\vec{E} \\
\mathrm{grad}\,\mathrm{div}\,\vec{B} - \nabla^2\vec{B} &= \mu_0\mu_r\epsilon_0\epsilon_r \cdot \left(-\frac{\partial^2 \vec{B}}{\partial t^2}\right) - \mu_0\mu_r\sigma\frac{\partial \vec{B}}{\partial t} \\
\nabla^2\vec{B} &= \frac{n^2}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} + \gamma\frac{\partial \vec{B}}{\partial t} \qquad \text{with } \gamma = \mu_0\mu_r\sigma
\end{aligned}$$

To solve this equation we use an Ansatz for a wave propagating in x-direction:

$$\vec{B}(\vec{r},t) = B_0 \underline{\hat{\mathsf{a}}}_z \exp i(kx - \omega t)$$

$$\vec{B}(\vec{r},t) = B_0 \underline{\hat{\mathsf{a}}}_z \exp(i(kx - \omega t))$$

$$\frac{\partial \vec{B}}{\partial x} = i\,k\vec{B} \qquad \frac{\partial^2 \vec{B}}{\partial x^2} = -k^2 \vec{B}$$

$$\frac{\partial \vec{B}}{\partial t} = i\,\omega\vec{B} \qquad \frac{\partial^2 \vec{B}}{\partial t^2} = -\omega^2 \vec{B}$$

Substitute these derivatives into the equation:

$$\begin{aligned}
\nabla^2 \vec{B} &= \frac{n^2}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} + \gamma \frac{\partial \vec{B}}{\partial t} \\
-k^2 \vec{B} &= -\frac{n^2}{c^2}\omega^2 \vec{B} - i\gamma\omega\vec{B} \\
k^2 &= \frac{n^2\omega^2}{c^2} + i\gamma\omega
\end{aligned}$$

$k^2 = n^2\omega^2/c^2 + i\gamma\omega$ is complex, thus k is also a complex number!

$$k = \sqrt{\frac{n^2\omega^2}{c^2} + i\gamma\omega}$$

Examine both terms for a typical good conductor with $\epsilon_r = \mu_r = 1$.

Then $n = 1$ $c = 3 \times 10^8$ m/s $\omega = 10^{14}$ rad/s

$\sigma = 10^8\,\Omega^-1\text{m}^-1 = 10^8$ A/(Vm)

$\gamma\omega = \mu_0\mu_r\sigma\omega = 4\pi \times 10^{-7}$ Vs/Am $\cdot\ 10^8$ A/(Vm) $\cdot\ 10^{14}$ rad/s $= 1.26 \times 10^{15}$ rad/m^2

$$\frac{n^2\omega^2}{c^2} = \frac{(10^{14}\ \text{rad/s})^2}{(3 \times 10^8\ \text{m/s})^2} = (3.3 \times 10^5\ \text{rad/m})^2 = 10^{11}\ \text{rad}^2/\text{m}^2$$

We find that usually $\gamma\omega >> (n^2\omega^2)/c^2$ thus

$$k^2 \approx i\omega\gamma$$

$$\rightarrow k = \sqrt{\frac{\omega\gamma}{2}} + i\sqrt{\frac{\omega\gamma}{2}} = k_0 + i\xi \qquad \text{with} \qquad k_0 = \xi = \sqrt{\frac{\omega\gamma}{2}}$$

The magnetic field now looks like this:

$$\begin{aligned}\vec{B} &= B_0\underline{\hat{\mathsf{a}}}_z \exp i((k_0 + i\xi)x - \omega t) \\ &= B_0\underline{\hat{\mathsf{a}}}_z \exp[i(k_0 x - \omega t) - \xi x] \\ &= B_0\underline{\hat{\mathsf{a}}}_z \exp[i(k_0 x - \omega t)] \exp[-\xi x]\end{aligned}$$

As the wave tries to move through the conductor, it gets attenuated!

4.34: Skin depth

$$\xi = \sqrt{\frac{\omega\gamma}{2}} = \sqrt{\frac{\omega\mu_0\sigma}{2}}$$

For $\sigma = 10^8 \text{A/(Vm)}$ and $\omega = 10^{14}\,\text{rad/s}$ we have

$$\begin{aligned} \xi &= \sqrt{10^{14}\,\text{rad/s} \cdot 2\pi \times 10^{-7}\,\text{Vs/Am} \cdot 10^{8}\,\text{A/(Vm)}} \\ &= \sqrt{6.3 \times 10^{15}\,\text{m}^{-2}} \\ &= 8 \times 10^{7}\,\text{m}^{-1} \end{aligned}$$

The penetration depth is defined as

$$\delta = 1/\xi = \sqrt{\frac{2}{\omega\mu_0\sigma}} \approx 0.125 \times 10^{-7}\ \text{m} = 12.5\,\text{nm}$$

The wave does not enter into the material but gets attenuated near the surface. We also call this the skin depth.

How thick is this skin?

At a depth of $4\delta \approx 50\,\text{nm}$ the original magnetic field has exponentially fallen to less than 1 % of its original value. This should be compared to the wavelength of the wave

$$\lambda = \frac{c}{f} = \frac{2\pi c}{\omega} = \frac{6.3 \cdot 3 \times 10^8\,\text{m/s}}{10^{14}\,\text{rad/s}} \approx 2 \times 10^{-5}\ \text{m} = 20\,\mu\text{m}$$

The skin depth is only a fraction of the wavelength.

4.35: Poynting Vector

Light carries energy. Now that we have established that light is a electromagnetic wave we can ask where the energy is stored and how it is transported.

We already found that energy is stored in the electric field of a plate capacitor:

$$U = \tfrac{1}{2}\int_V \vec{E}\cdot\vec{D}\,dV$$

Thus we will hopefully start from Maxwell's equations to find the flow of energy in a general case. Intuitively we expect that the energy should flow perpendicular to $\vec{E}$ and $\vec{B}$, and energy should be stored in both fields $\vec{E}$ and $\vec{B}$.

Start from

$$\begin{aligned} \text{div}\vec{D} &= \rho \qquad \text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \text{div}\vec{B} &= 0 \qquad \text{curl}\,\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \end{aligned}$$

Assume that the current density is produced by moving charges $\vec{j} = \rho \cdot \vec{v}$. Then the energy dissipated at any point is $\vec{j} \cdot \vec{E} = \rho\vec{v} \cdot \vec{E}$ and in the volume V the energy dissipated is

$$\int_V \rho\vec{v} \cdot \vec{E}\, dV = \int_V \vec{j} \cdot \vec{E}\, dV$$

But $\vec{j} \cdot \vec{E}$ can be rewritten with Maxwell's equations

$$\begin{aligned} \text{curl}\vec{H} &= \frac{\partial\vec{D}}{\partial t} + \vec{j} \quad \Rightarrow \quad \vec{j} = \text{curl}\vec{H} - \frac{\partial\vec{D}}{\partial t} \\ \vec{j} \cdot \vec{E} &= \vec{E} \cdot \vec{j} = \vec{E} \cdot (\text{curl}\,\vec{H} - \frac{\partial\vec{D}}{\partial t}) \end{aligned}$$

Examine $\vec{E} \cdot (\vec{\nabla} \times \vec{H})$ more closely:

For any vector fields $\vec{A}$ and $\vec{B}$ we have

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$$

Exercise: confirm this by explicitly calculating both sides in cartesian coordinates.

We can use this by taking the divergence of $\vec{E} \times \vec{H}$:

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H})$$

to find

$$\vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \cdot (\vec{E} \times \vec{H})$$

Thus

$$\begin{aligned} \vec{j} \cdot \vec{E} &= \vec{E} \cdot (\vec{\nabla} \times \vec{H}) - \vec{E}\frac{\partial \vec{D}}{\partial t} \\ \vec{j} \cdot \vec{E} &= \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \cdot (\vec{E} \times \vec{H}) - \vec{E}\frac{\partial \vec{D}}{\partial t} \end{aligned}$$

We use $\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t$ to get

$$\vec{j} \cdot \vec{E} = -\vec{H}\frac{\partial \vec{B}}{\partial t} - \vec{E}\frac{\partial \vec{D}}{\partial t} - \vec{\nabla} \cdot (\vec{E} \times \vec{H})$$

This is valid at each point. Let us now consider a small volume V bounded by the closed surface S:

$$\int_V \vec{j} \cdot \vec{E}\, dV = -\int_V \left(\vec{H}\frac{\partial \vec{B}}{\partial t} + \vec{E}\frac{\partial \vec{D}}{\partial t} \right) dV - \int_V \vec{\nabla} \cdot (\vec{E} \times \vec{H})\, dV$$

The last term is converted to an integral over the surface using Gauss' Law:

$$\int_V \vec{j} \cdot \vec{E}\, dV = -\int_V \left(\vec{H}\frac{\partial \vec{B}}{\partial t} + \vec{E}\frac{\partial \vec{D}}{\partial t} \right) dV - \int_S (\vec{E} \times \vec{H})\, d\vec{S}$$

The term on the lefthand side is the energy dissipated by ohmic currents $\vec{j} = \sigma \vec{E}$:

$$\int_V \vec{j}\vec{E}\, dV = \int_V \sigma E^2\, dV = \int_V \frac{j^2}{\sigma}\, dV$$

The righthand side has two terms.

$$-\int_V \left(\vec{H}\frac{\partial \vec{B}}{\partial t} + \vec{E}\frac{\partial \vec{D}}{\partial t}\right)\, dV$$

can be interpreted as the decrease of energy stored in the fields in the volume V if we take the total energy to be

$$\begin{aligned} U &= \tfrac{1}{2}\int_V \left(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D}\right)\, dV \\ \frac{\partial U}{\partial t} &= \tfrac{1}{2}\int_V \left(\frac{\partial \vec{H}}{\partial t}\vec{B} + \vec{H}\frac{\partial \vec{B}}{\partial t} + \frac{\partial \vec{E}}{\partial t}\vec{D} + \vec{E}\frac{\partial \vec{D}}{\partial t}\right)\, dV \end{aligned}$$

But in LIH media $\vec{B} = \mu_0\mu_r\vec{H}$ thus

$$\frac{\partial \vec{H}}{\partial t}\vec{B} = \frac{\partial}{\partial t}(\frac{1}{\mu_0\mu_r}\vec{B})\cdot\mu_0\mu_r\vec{H} = \frac{\partial \vec{B}}{\partial t}\vec{H}$$

And similarily

$$\vec{D}\frac{\partial E}{\partial t} = \vec{E}\frac{\partial \vec{D}}{\partial t}$$

Thus

$$\begin{aligned}
\frac{\partial U}{\partial t} &= \tfrac{1}{2}\int_V \left(2\vec{H}\frac{\partial \vec{B}}{\partial t} + 2\vec{E}\frac{\partial \vec{D}}{\partial t}\right) dV \\
&= \int_V \left(\vec{H}\frac{\partial \vec{B}}{\partial t} + \vec{E}\frac{\partial \vec{D}}{\partial t}\right) dV
\end{aligned}$$

The last term must represent the flow of energy into the volume (i.e. through the surface) as expected.

$$-\int_S (\vec{E} \times \vec{H}) \cdot d\vec{S} \quad \text{is} \quad \text{the flow in}$$

$$\int_S (\vec{E} \times \vec{H}) \cdot d\vec{S} \quad \text{is} \quad \text{the flow of energy } \textit{out} \text{ of the volume}$$

4.36: The Poynting Theorem

The vector $\vec{N} = \vec{E} \times \vec{H}$ is called the Poynting vector. It is perpendicular to both $\vec{E}$ and $\vec{H}$ (and thus $\vec{B}$) as required.

We write the Poynting theorem (which is a glorified name for conservation of energy):

$$\int_S \vec{N} \cdot d\vec{S} \quad = \quad -\int_V \left(\vec{H} \frac{\partial \vec{B}}{\partial t} + \vec{E} \frac{\partial \vec{D}}{\partial t} \right) dV \qquad - \qquad \int_V \vec{j} \cdot \vec{E} \, dV$$

Energy flow out of volume	Decrease of energy stored in the fields within volume	Ohmic Losses (generate heat)

4.37: Average power transmitted

EM waves have an infinite range of wavelengths and frequencies with $\lambda f = c$.

If we talk about energy transport it makes sense to average the energy over at least one period and examine the steady flow of energy ($\equiv$ average power).

$$< \vec{N} >= \frac{1}{T} \int_0^T \vec{N} dt$$

The easiest way to see how this works is by example: An EM wave is given by the time dependent $\vec{E}$ and $\vec{B}$ fields

$$\begin{aligned} \vec{E}(\vec{r}, t) &= E_0 \, \underline{\hat{\mathsf{a}}}_x \sin k(z - ct) \\ \vec{B}(\vec{r}, t) &= \frac{E_0}{c} \underline{\hat{\mathsf{a}}}_y \sin k(z - ct) \end{aligned}$$

This gives the Poynting vector

$$\vec{N}(\vec{r},t) = \frac{E_0^2}{\mu_0 c}\sin^2 k(z-ct)\underline{\hat{\mathsf{a}}}_x \times \underline{\hat{\mathsf{a}}}_y = \frac{E_0^2}{\mu_0 c}\sin^2 k(z-ct)\underline{\hat{\mathsf{a}}}_z$$

$\vec{N}$ is perpendicular to $\vec{E}$ and $\vec{B}$.

Units:

$$\left[\frac{E_0^2}{\mu_0 c}\right] = \frac{\mathrm{VA}}{\mathrm{m}^2} = \frac{\mathrm{J/C} \cdot \mathrm{C/s}}{\mathrm{m}^2} = \frac{\mathrm{J}}{\mathrm{sm}^2} = \frac{\mathrm{W}}{\mathrm{m}^2} \quad = \frac{\text{Power}}{\text{Area}} \quad \text{as expected}$$

But the instantaneous power oscillates between 0 and $E_0^2/\mu_0 c$ twice per period, for visible light thats 2×10^{15} times per second. The time average is

$$
\begin{aligned}
<\vec{N}> &= \frac{1}{T}\int_0^T \vec{E} \times \vec{H} dt \\
&= \frac{1}{T}\int_0^T \frac{E_0 \,\underline{\hat{\mathsf{a}}}_z}{\mu_0 c} \sin^2 k(z-ct) dt \\
&= \frac{E_0 \,\underline{\hat{\mathsf{a}}}_z}{\mu_0 c}\frac{1}{T}\int_0^T \sin^2 k(z-ct) dt
\end{aligned}
$$

But we know that the average of $\sin^2$ over one full period is $1/2$. Thus

$$
<\vec{N}> = \frac{E_0^2 \,\underline{\hat{\mathsf{a}}}_z}{2\mu_0 c}
$$

This is the average power transmitted per unit area.

4.38: Example 1

$$\vec{E} = E_0\,\underline{\hat{\mathsf{a}}}_x \sin k(z - ct)$$
$$\vec{B} = \frac{E_0}{c}\,\underline{\hat{\mathsf{a}}}_y \sin k(z - ct) \qquad \vec{H} = \frac{1}{\mu_0}\vec{B}$$

Find the average flux of energy through a square area with sides in the $z = 0$ plane in vacuum:

$$< \vec{N} > = < \vec{E} \times \vec{H} > = \frac{1}{\mu_0} < \vec{E} \times \vec{B} >$$
$$< \vec{N} > = \frac{E_0^2}{2\mu_0 c}\,\underline{\hat{\mathsf{a}}}_z$$

since $\vec{N}$ does not depend on x or y we can easily evaluate

$$\begin{aligned} < P > &= = \int_S < \vec{N} > \cdot d\vec{S} \\ &= \int_0^a \int_0^a < \vec{N} > \cdot dx dy\,\underline{\hat{\mathsf{a}}}_z \\ &= \int_0^a \int_0^a \frac{E_0^2}{2\mu_0 c}\,\underline{\hat{\mathsf{a}}}_z \cdot \underline{\hat{\mathsf{a}}}_z dx dy = \frac{E_0^2 a^2}{2\mu_0 c} \end{aligned}$$

For $E_0 = 0.1$ V/m and $a = 1$ m we get

$$< P >= \frac{(0.1\,\text{V/m})^2 \cdot (1\,\text{m})^2}{2 \cdot 4\pi \times 10^{-7} \frac{\text{Vs}}{\text{Am}} \cdot 3 \times 10^8 \ \text{m/s}} = \frac{10^{-2}}{240\pi}\ \text{W}$$

At this point we can also tie in with your waves course. it is very convenient to write $\vec{E}$ and $\vec{B}$ in complex form

$$\begin{aligned} \vec{E}(\vec{r}, t) &= E_0\, \underline{\hat{\mathsf{a}}}_x \exp ik(z - ct) \\ \vec{B}(\vec{r}, t) &= \frac{E_0}{c}\, \underline{\hat{\mathsf{a}}}_y \exp ik(z - ct) \end{aligned}$$

The real and imaginary parts of these are

$$\begin{aligned} \text{Re}\, \vec{E} &= E_0\, \underline{\hat{\mathsf{a}}}_x \cos k(z - ct) \qquad \text{Im}\, \vec{E} = E_0\, \underline{\hat{\mathsf{a}}}_x \sin k(z - ct) \\ \text{Re}\, \vec{B} &= \frac{E_0}{c}\, \underline{\hat{\mathsf{a}}}_y \cos k(z - ct) \qquad \text{Im}\, \vec{B} = \frac{E_0}{c}\, \underline{\hat{\mathsf{a}}}_y \sin k(z - ct) \end{aligned}$$

If we take $\vec{N} = \vec{E} \times \vec{H}$ we end up with

$$\vec{N} = \frac{E_0^2}{\mu_0 c} \underline{\hat{\mathsf{a}}}_z \exp ik(z - ct)$$

and we have a complex flow of energy. That becomes difficult to visualise. However, if we evaluate

$$\begin{aligned} &\tfrac{1}{2}(\vec{E} \times \vec{H}^*) \quad \text{where} \quad \vec{H}^* \text{ is the complex conjugate of} \quad \vec{H} \\ &= \frac{E_0^2}{2\mu_0 c} \exp[ik(z - ct)] \cdot \exp[-ik(z - ct)] \\ &= \frac{E_0^2}{2\mu_0 c} \end{aligned}$$

we immediately end up with our earlier definition of average power.

Thus we can adopt the definition

$$< \vec{N} > = \tfrac{1}{2}(\vec{E} \times \vec{H}^*)$$

as the average power transported per unit area.

4.39: Summary

The index of refraction is

$$n = \sqrt{\epsilon_r \mu_r}$$

The skin depth on a conductor is

$$d = 1/\delta = \sqrt{\frac{2}{\omega \mu_0 \sigma}}$$

The Poynting vector is

$$\vec{N} = \vec{E} \times \vec{H}$$

The Poynting theorem:

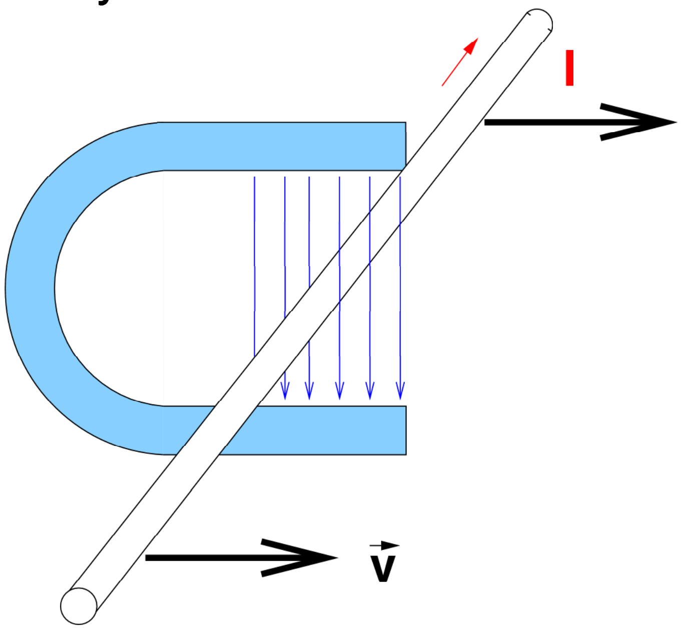
$$\int_S \vec{N} \cdot d\vec{S} \quad = \quad -\int_V \left(\vec{H} \frac{\partial \vec{B}}{\partial t} + \vec{E} \frac{\partial \vec{D}}{\partial t} \right) dV \quad - \quad \int_V \vec{j} \cdot \vec{E} \, dV$$

Energy flow out of volume	Decrease of energy stored in the fields within volume	Ohmic Losses in the volume

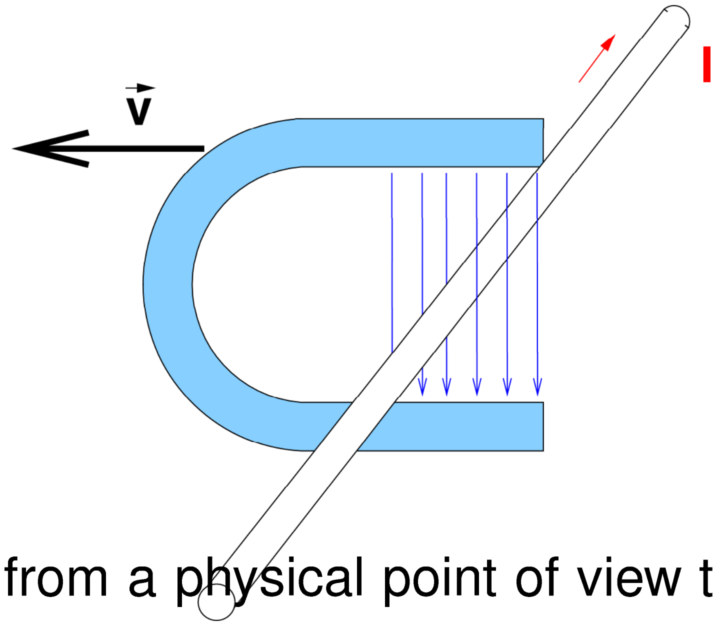
4.40: Electromagnetism and Special Relativity

In his 1905 paper “Electrodynamics of moving bodies” Einstein laid the foundations for special relativity. It is all already contained in Maxwell’s equations. In this last bit of the course we will see how special relativity and Maxwell’s equations relate to each other.

Consider a conductor moving relative to a horseshoe magnet. We can treat this either as a stationary magnet and a moving conductor or as a moving magnet and a stationary conductor.



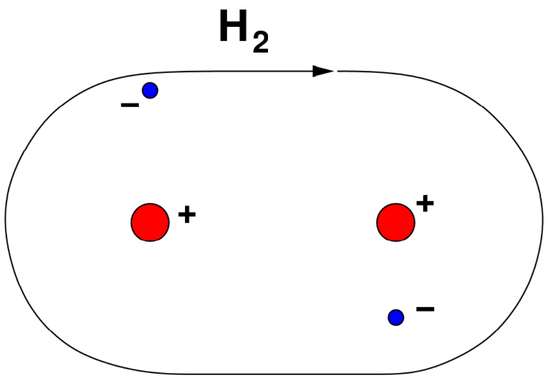
A) moving conductor, stationary magnet
The conductor cuts through the $\vec{B}$, thus the charges moving with the conductor experience a Lorentz force and a current flows. In the picture the current flows into the page.

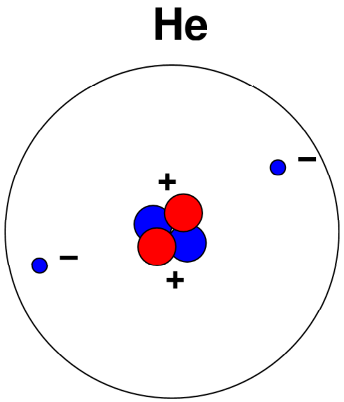


B) stationary conductor, moving magnet
The time dependent magnetic field creates an electric field at the position of the conductor which moves the electrons inside it creating a current. The net effect is the same, current moves into the page.

But from a physical point of view these are very different processes governed in one case by the Lorentz force (A) and in the other by an induced electric field (B). However, it is clear that only the relative motion of magnet and conductor is relevant, thus the difference in physical interpretation must be ascribed to the different observers: in (A) the observer is at rest relative to the magnet, in (B) s/he is at rest relative to the conductor.

In a second step we examine the relativistic behaviour of charge. Does an observer in a moving frame measure a different charge on an electron than a stationary observer? Consider a He-atom and a H_2 molecule. The electrons in both move at roughly the same speed. The two protons move at vastly different speeds. In H_2 they can be treated as being at rest. In the He nucleus they are confined to a space with $3fm$ diameter.





Heisenberg's uncertainty principle gives us a minimum momentum:

$$\begin{aligned} \Delta x \cdot \Delta p &\geq \hbar \\ \Delta p &\geq \frac{\hbar}{\Delta x} \end{aligned}$$

And a minimum velocity of

$$\begin{aligned} \Delta v &\geq \frac{\hbar}{\Delta x \cdot m} \\ \Delta v &\geq \frac{6.6 \times 10^{-34} Js}{1.67 \times 10-27 kg \cdot 3 \times 10^{-15} m} = 1.3 \times 10^{8} m/s \end{aligned}$$

This is a significant fraction of the speed of light. Both the He atom and the H_2 molecule are electrically neutral and we are forced to conclude that charge is relativistically invariant: The charge on an electron is measured as $-e$ by any observer, regardless of the observer's state of motion relative to the electron.

Charge is relativistically invariant!

Thirdly, we already found in our discussions of Faraday's Law that an electric field measured in two frames of reference K at rest and K' moving with a small velocity $\vec{v}$ relative to K is given by $\vec{E}$ in K and

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$$

in K'. At large velocities this will be slightly modified.

4.41: Reminder Special Relativity

Frames K and K' have parallel axes. At $t = 0$ the axes coincide. Frame K' moves with velocity $\vec{v} = v\,\underline{\hat{\mathsf{a}}}_x$ in positive x-direction reltative to frame K.

We define $\gamma = 1/\sqrt{1 - v^2/c^2}$.

In frame K we measure $\vec{r}, t, \vec{E}, \vec{B}$, in K' we measure $\vec{r}', t', \vec{E}', \vec{B}'$.

Between K and K' we have the Lorentz transformations:

$$
\begin{aligned}
x' &= \gamma(x - vt) \\
y' &= y \\
z' &= z \\
t' &= \gamma(t - \frac{vx}{c^2})
\end{aligned}
$$

Lorentz contraction:

If L_x is the length in x-direction of an object stationary in K, its length L'_x in K' is

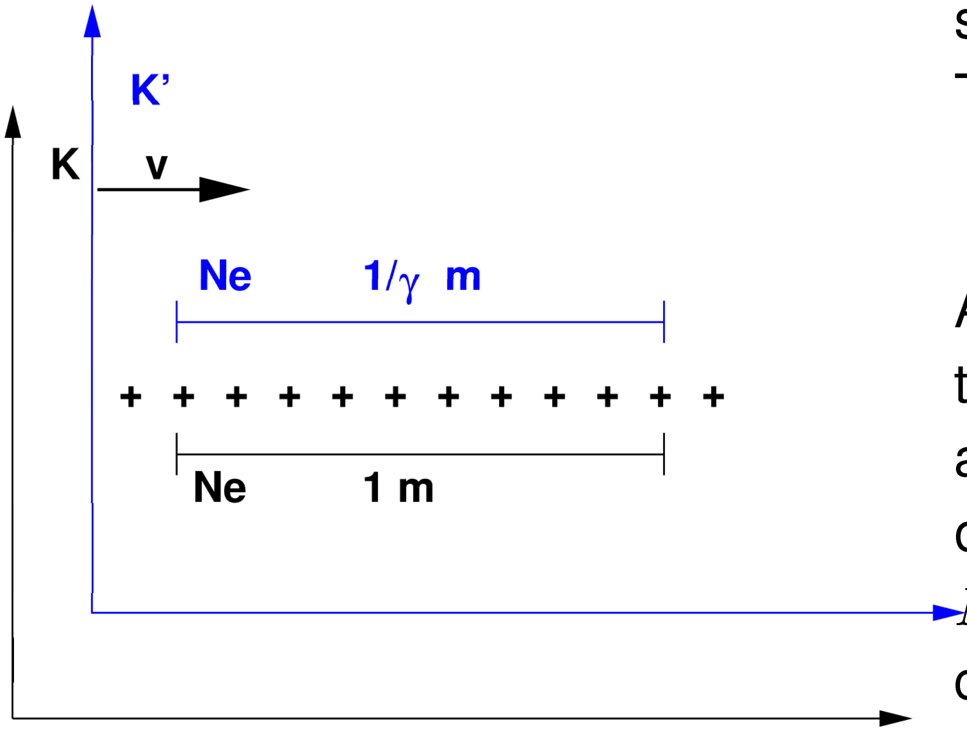
$$L'_x = (1/\gamma)\, L_x$$

Time dilation:

If T is the time interval between two events *at the same place* in K, the time interval between the same events in K' is

$$T' = \gamma\, T$$

4.42: Charge density



First we transform a line charge density. In frame K we have a number of positive charges $+e$ distributed along an infinite straight line with N charge per unit length. This is a linear charge density

$$\lambda = Ne$$

An observer moving with velocity v parallel to the line sees the same charges but disagrees about the distance between them, because of the Lorentz contraction. S/he thus sees N charges per length $(1m/\gamma)$ giving a larger charge density

$$\lambda' = \gamma Ne$$

Take a test charge Q a distance a from a neutral wire. Q is at rest in K. In K the wire carries a current I. The wire is neutral and therefore carries an equal number of positive and negative charges per unit length.

Assume all negative charges are at rest in K and the current is created by all positive charges moving with speed v.

Then the current is $I = Nev$ and the magnetic field created by this current at a distance a from the wire is

$$B = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 Nev}{2\pi r}$$

If the charge Q now moves parallel to the wire with velocity v at that distance r it experiences the Lorentz force toward the wire of magnitude

$$F = QvB = Q\frac{\mu_0 Nev^2}{2\pi r}$$

This is a purely magnetic force.

Now analyse this situation from frame K' where Q and the positive charges are at rest.

Here the negative charges in the wire move in the opposite direction with velocity $-v$, thus their linear charge density appears to be

$$\lambda'_- = -\gamma N e$$

The positive charges are now at rest and their charge density is now reduced to

$$\lambda'_+ = \frac{Ne}{\gamma}$$

Thus the observer in K' sees a net charge in the wire

$$\begin{aligned} \lambda' &= \lambda'_+ + \lambda'_- \\ &= \gamma N e(\frac{1}{\gamma^2} - 1) = -\gamma N e \frac{v^2}{c^2} \\ &= \frac{-\gamma I v}{c^2} \end{aligned}$$

The distance r is perpendicular to the wire. Therefore it is not Lorentz contracted and we have a purely electric force between a point charge and a line charge.

The electric field of a line charge was

$$|E'| = \frac{\lambda}{2\pi\epsilon_0 r} = \frac{\gamma N e v^2}{2\pi\epsilon_0 r c^2}$$

But $c^2 = (1/\mu_0\epsilon_0)$ so

$$E' = \frac{\gamma N e v^2 \mu_0}{2\pi r} = \gamma v B$$

And the force is $F' = QE'$

$$F' = \gamma Q v B = \gamma F$$

The purely magnetic Loretnz force in frame K turns into a pure Coulomb force in frame K'. Furthermore the two observers also disagree about the magnitude of the force

$$F' = \gamma F$$

However, the motion of the charge Q must not depend on the observer.

We can reconcile the situation if we remember not only Lorentz contraction but time dilation. A force is a change of momentum per unit time.

$$F = \frac{dp}{dt}$$

$$\begin{aligned} \text{In frame } K: \qquad F &= \frac{dp}{dt} \\ \text{In frame } K': \qquad F' &= \frac{dp'}{dt'} \end{aligned}$$

Since the force points toward the wire and therefore perpendicular to the motion of K' the same is true for the momentum. However, the perpendicular components of p are unchanged in Lorentz transformations, but time is not.

Both observers agree that the charge obtains momentum $dp' = dp$. But in K it takes time dt, while an observer in K' thinks it takes time dt'. For him the clock in K goes slow so he thinks $dt = \gamma dt'$

$$F' = \frac{dp'}{dt'} = \frac{dp}{dt/\gamma} = \gamma F$$

This means that both observers agree on the physical consequences for the motion of the charge Q.

This is the essence of special relativity.

We shall now investigate the transformation of Maxwell's equations under Lorentz transformations. We also need to see how the differentials transform:

In components we get

$$\begin{aligned} E'_x &= E_x \\ E'_y &= \gamma(E_y - vB_z) \\ E'_z &= \gamma(E_z + vB_y) \\ B'_x &= B_x \\ B'_y &= \gamma(B_y + \frac{v}{c^2}E_z) \\ B'_z &= \gamma(B_z - \frac{v}{c^2}E_y) \end{aligned}$$

$$\frac{\partial}{\partial x'} = \gamma(\frac{\partial}{\partial x} - \frac{v}{c^2}\frac{\partial}{\partial t})$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial t'} = \gamma(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x})$$

In K' $\qquad \vec{\nabla}' \cdot \vec{B}' = 0$

$$
\begin{aligned}
\frac{\partial}{\partial x'}B'_x + \frac{\partial}{\partial y'}B'_y + \frac{\partial}{\partial z'}B'_z &= \gamma\frac{\partial B_x}{\partial x} - \gamma\frac{v}{c^2}\frac{\partial B_x}{\partial t} + \gamma\frac{\partial B_y}{\partial y} + \gamma\frac{v}{c^2}\frac{\partial E_z}{\partial y} + \gamma\frac{\partial B_z}{\partial z} - \gamma\frac{v}{c^2}\frac{\partial E_y}{\partial z} \\
&= \gamma(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}) - \gamma\frac{v}{c^2}(\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z})
\end{aligned}
$$

But in K we have

$$\vec{\nabla} \cdot \vec{B} = 0$$

and

$$\text{curl}\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

Therefore in K' we find

$$\vec{\nabla}' \cdot \vec{B}' = 0$$

The same goes for all other Maxwell equations.

But if all Maxwell equations are the same in both frames of reference, the two observers must also derive identical wave equations:

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \qquad and \qquad \nabla'^2 \vec{E'} = \frac{1}{c^2}\frac{\partial^2 \vec{E'}}{\partial t'^2}$$

This means automatically that they measure the same speed of light, regardless of their state of relative motion.

Einstein's genius was that he saw these arguments in reverse. He asked: "Given a set of Maxwell's equations which are invariant under the change of frame of reference, what are the transformations that govern this change?"

It can be shown that the only transformations in this case are the Lorentz transformations. Special Relativity was born.