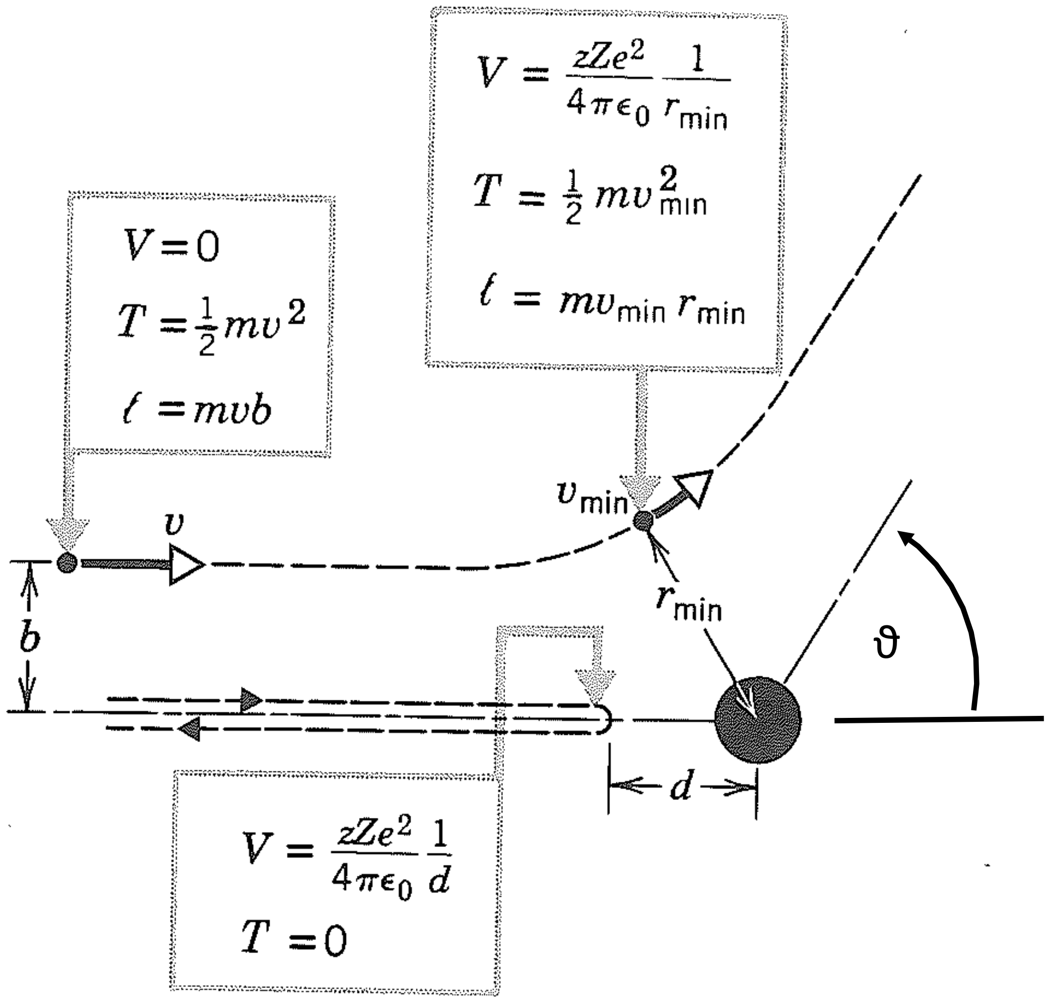
# Rutherford Scattering for pedestrians

See Krane, pp 396-399

# Rutherford Scattering




Characteristic:

Impact parameter **b**
kinetic energy of projectile
scattering angle **$\vartheta$**

Use:
Conservation of
- Energy
- Angular Momentum
- Momentum

# Some relations

Distance of closest approach in head-on collision:

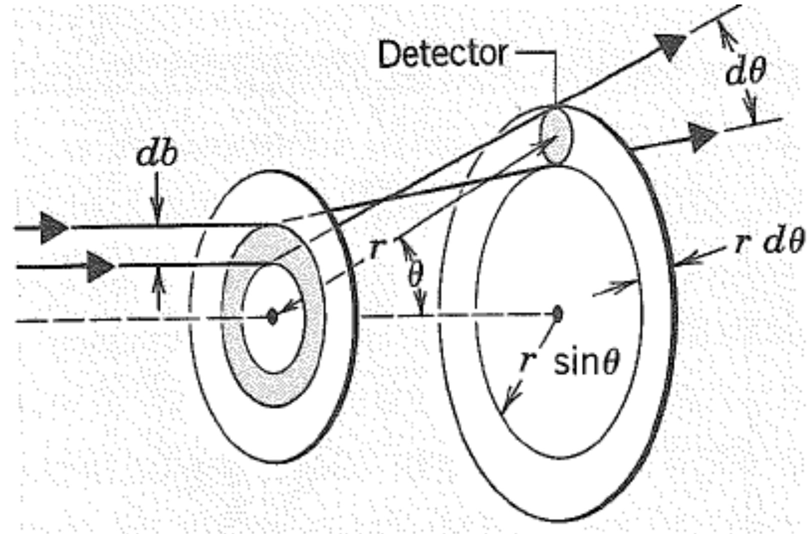
$$\tfrac{1}{2} m v_0^2 = \frac{1}{4\pi\varepsilon_0} \frac{zZe^2}{d}$$

Energy balance anywhere:

$$\tfrac{1}{2} m v_0^2 = \tfrac{1}{2} m v^2 + \frac{1}{4\pi\varepsilon_0} \frac{zZe^2}{r}$$

# Particle flux

Incident particles with impact parameter between **b** and **b+db** go into a cone shell with opening angles between **ϑ** and **ϑ+dϑ**



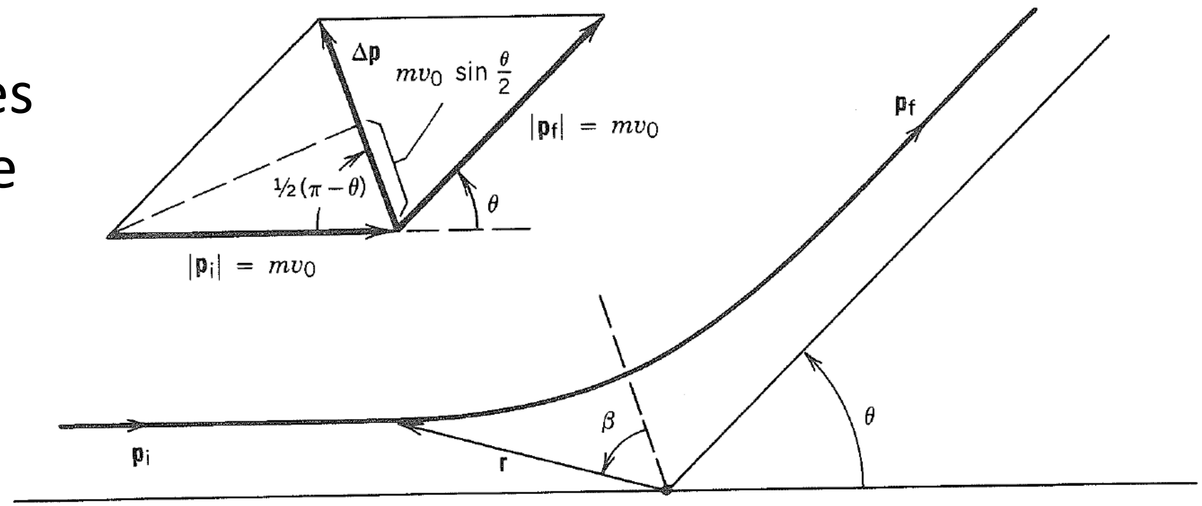

The fraction of particles going through this ring is

$$\Delta f = nx \cdot (2\pi b \Delta b)$$

(Δ used instead of d for clarity only!)

# Momentum balance

Momentum only changes direction, not magnitude (elastic scattering!)




The change in momentum is then easily found from the geometry:

$$|\Delta p| = 2mv_0 \sin\frac{\vartheta}{2}$$

# Newton's 2nd Law:

Use Coulomb Force to get change in Momentum:

$$\Delta p = \int dp = \int F\,dt = \frac{zZe^2}{4\pi\varepsilon_0}\int \frac{dt}{r^2}\cos\beta(t)$$

The angle β is defined on the picture and changes with time. It gives the tangential direction to the target at all times.

$$\vec{v} = \frac{d\vec{r}}{dt}\hat{r} + r\frac{d\vec{\beta}}{dt}\hat{\beta}$$

velocity = radial velocity + tangential velocity

# Angular Momentum

Only tangential velocity contributes!

$$\left|\vec{L}\right| = \left|\vec{r} \times m\vec{v}\right| = mr^2 \frac{d\beta}{dt}$$

But at great distances we have

$$\left|\vec{L}\right| = mv_0 b$$

So we can relate *b* and β:

$$mv_0 b = mr^2 \frac{d\beta}{dt} \qquad \text{or} \qquad \frac{dt}{r^2} = \frac{d\beta}{v_0 b}$$

# Integrate

Now we can solve the integral:

$$\Delta p = \int dp = \int F\,dt = \frac{zZe^2}{4\pi\varepsilon_0}\int \frac{dt}{r^2}\cos\beta(t) = \frac{zZe^2}{4\pi\varepsilon_0}\int \frac{d\beta}{v_0 b}\cos\beta$$

$$= \frac{zZe^2}{4\pi\varepsilon_0 v_0 b}\int_{-(\frac{\pi}{2}-\frac{\vartheta}{2})}^{+(\frac{\pi}{2}-\frac{\vartheta}{2})} \cos\beta\,d\beta = \frac{zZe^2}{2\pi\varepsilon_0 v_0 b}\cos\frac{\vartheta}{2}$$

$$\Rightarrow \Delta p = \frac{zZe^2}{2\pi\varepsilon_0 v_0 b}\cos\frac{\vartheta}{2}$$

# Combine:

$$\Delta p = \frac{zZe^2}{2\pi\varepsilon_0 v_0 b}\cos\frac{\vartheta}{2} \qquad \Delta p = 2mv_0\sin\frac{\vartheta}{2} \qquad \tfrac{1}{2}mv_0^2 = \frac{1}{4\pi\varepsilon_0}\frac{zZe^2}{d}$$

$$\frac{zZe^2}{2\pi\varepsilon_0 v_0 b}\cos\frac{\vartheta}{2} = 2mv_0\sin\frac{\vartheta}{2} = \frac{1}{\pi\varepsilon_0}\frac{zZe^2}{dv_0}\sin\frac{\vartheta}{2}$$

$$\Leftrightarrow \frac{1}{2b}\cos\frac{\vartheta}{2} = \frac{1}{d}\sin\frac{\vartheta}{2}$$

$$\Leftrightarrow b = \frac{d}{2}\cot\frac{\vartheta}{2}$$

This is the relation between scattering angle and impact parameter we were after!

# Bring it together:

$$df = nx \cdot (2\pi b db)$$

$$b = \frac{d}{2}\cot\frac{\vartheta}{2} \Rightarrow db = -\frac{d}{4}\csc^2\frac{\vartheta}{2}d\vartheta$$

This gives the fraction of the outgoing particle flux on the ring between b and b+db as

$$df = nx \cdot (2\pi b db) = nx\pi\frac{d^2}{4}\cot\frac{\vartheta}{2}\csc^2\frac{\vartheta}{2}d\vartheta$$

We can turn this into a cross section:

Cross Section=$\text{Flux}_{\text{out}}/(\text{Flux}_{\text{in}} \times N_{\text{target}})$

# Cross Section

Flux in: $I_a$ Flux out: $I_a df$ $N_{target} = nx$

Scattering into a solid angle element dΩ:

$d\Omega = \sin\vartheta \, d\vartheta \, d\phi = 2\pi \sin\vartheta \, d\vartheta$ (axial symmetry!)

$= \pi \sin \vartheta/2 \cos \vartheta/2 \, d\vartheta$

$$\frac{I_a df}{d\Omega}\frac{1}{I_a nx} = \frac{I_a nx\pi}{I_a nx\pi \sin{}^{\vartheta}\!/\!_2 \cos{}^{\vartheta}\!/\!_2 \, d\vartheta}\frac{d^2}{4}\frac{\cos{}^{\vartheta}\!/\!_2}{\sin^3{}^{\vartheta}\!/\!_2}d\vartheta$$

$$\frac{d\sigma}{d\Omega} = \frac{d^2}{4}\frac{1}{\sin^4{}^{\vartheta}\!/\!_2} = \left(\frac{zZe^2}{4\pi\varepsilon_0}\right)^2\left(\frac{1}{4E^2}\right)\frac{1}{\sin^4{}^{\vartheta}\!/\!_2}$$